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ABSTRACT

This student textbook is designed to incorporate programable calculators in grade 12 mathematics. The seven chapters contained in this document are: (1) Using Calculators in Mathematics; (2) Sequences, Series, and Limits; (3) Iteration, Mathematical Induction, and the Binomial Theorem; (4) Applications of the Fundamental Counting Principle; (5) Polynomial Functions; (6) Exponents and Logarithms; and (7) Trigonometric Functions. Each chapter is further subdivided into topic sections. Each section concludes with a set of calculator-oriented exercises geared towards the material covered. (MP)

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STUDENT TEXT
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USING CALCULATORS IN
MATHEMATICS 12

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CHAPTER 1. USING CALCULATORS IN MATHEMATICS

In this chapter you will become familiar with different kinds of calculating languages and you will learn how to compute with algebraic, AH, RPN, and arithmetic calculator logics. You will also learn simple programming and how to translate a verbal algorithm into calculator, microprocessor, or computer steps.

1.1 Order of Operations

Communication of ideas is important in mathematics. The reader of mathematics must understand what the writer of mathematics means. For this reason we adopt rules for writing and reading that are generally accepted. For example, when we write

$$5 + 2 \times 3 \quad .$$

we want all readers to interpret what we have written in the same way. Of the two choices

(a)		(b)	
	$5 + 2 \times 3$		$5 + 2 \times 3$
	7×3		$5 + 6$
	21	,	11

you would probably choose (b), answer 11, because you recall rules for order of operations. Unfortunately a younger brother or sister in elementary school or an adult who has not studied school mathematics (or has forgotten it) would probably choose (a). Thus rules designed to improve communication sometimes fail. We will review those rules and see how calculators force us once again to watch our step.

ORDER OF OPERATIONS RULE. Apply operations in the following order:

(1) within parentheses
(2) exponentiation (powers and roots)
(3) multiplication and division*
(4) addition and subtraction

Students sometimes remember this rule by the mnemonic:

Please Enter, My Dear Aunt Sally (for parentheses, exponents, multiplication, division, addition, subtraction). Only in the absence of rule priorities do you calculate *left* to *right*.

EXAMPLE 1.1-1 Evaluate $2 \cdot 4^2 - 6 \div \frac{\sqrt{9}}{7} + 5 \cdot 8$

Solution: $2 \cdot 4^2 - 6 \div \frac{\sqrt{9}}{7} + 5 \cdot 8$

$2 \cdot 16 - 6 \div \frac{3}{7} + 5 \cdot 8$ exponentiation

$32 - 14 + 40$ multiplication and division

58 addition and subtraction

EXAMPLE 1.1-2 Evaluate $\frac{2 \cdot 5 + 3 \cdot 4}{10 \cdot 2 - 3^2}$

Solution: $\frac{2 \cdot 5 + 3 \cdot 4}{10 \cdot 2 - 3^2}$,

$\frac{2 \cdot 5 + 3 \cdot 4}{10 \cdot 2 - 9}$ exponentiation

---

* In order to avoid rare instances where confusion might arise, some authors insist upon multiplication before division. We do not adopt that convention in this text.

$\frac{10 + 12}{20 - 9}$ multiplication. Note that the fraction bar (vinculum) plays a role as parentheses.* Thus numerator and denominator are simplified before division.

$\frac{22}{11}$ addition and subtraction

2 division

These examples have been worked out in detail. In practice many of these steps would be skipped. For example the second solution might be recorded as

$$\frac{2 \cdot 5 + 3 \cdot 4}{10 \cdot 2 - 3^2} = \frac{22}{11} = 2$$

Exercise Set 1.1

1 - 8 Evaluate:

(1) $2 \cdot 3 + 6$ (2) $2(3 + 6)$
(3) $6 + 2 \cdot 3$ (4) $(6 + 2)3$
(5) $7 + 2 \cdot 5$ (6) $(7 + 2)5$
(7) $2 \cdot 5 + 7$ (8) $2(5 + 7)$

9 - 16 Some writers use parentheses as "insurance" to guarantee that readers will calculate in the desired order. When it is possible in each of the following, write an equivalent expression without parentheses.

(9) $(ab) + (cd)$ (10) $(a + b)(c + d)$

(11) $(\frac{a}{b} \div \frac{c}{d}) \times \frac{e}{f}$ (12) $\frac{a}{b} \div (\frac{c}{d} \times \frac{e}{f})$

---

* Another example of this usage is in roots like $\sqrt{x + y}$. The root symbol is $\sqrt{\ }$ and the bar is a grouping symbol. In Europe $\sqrt{x + y}$ is often written $\sqrt{\ }(x + y)$.

(13) $a[b + c(d + e)]$ (14) $\{[(a + b) + c] - d\}$

(15) $\{[(ab) + c]d\} + e$ (16) $\dfrac{\sqrt{(a + b)}}{[(cd) - e]}$

17. For each of the exercises 9 - 16, evaluate (a) the original expression and (b) your simplified expression for the values $a = 6$, $b = 3$, $c = 4$, $d = 2$, $e = 7$, $f = -1$.

18 - 24. Notice in the following exercises how order makes no difference in exercises involving addition and subtraction, but seems to in exercises involving multiplication and division. Evaluate:

(18) $2 - 3 + 5$ (19) $2 + 5 - 3$

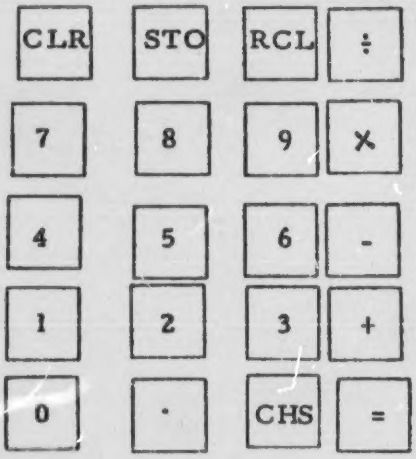
(20) $-3 + (5 + 2)$ (21) $2 \div 10 \times 5$ Be careful!

(22) $2 \times 5 \div 10$ (23) $2 \div (10 \times 5)$

1.2 Calculating Logic: Algebraic with Memory

The calculator or computer user must learn how to process numbers on the specific instrument he is using. This is important because of differences among calculating devices. In this section and the next three, we introduce several common calculating "logics". Most calculators operate by one of them or by a minor variation. A user tends to become accustomed to the logic of his machine and to prefer it. Indeed, each has certain advantages which we will consider. Even if you will be using a calculating device with a particular logic system, it is important to know how the others work. Who knows what kind you'll be using next?

ALGEBRAIC LOGIC



Algebraic logic is a common calculator logic. The figure displays a keyboard for a simple algebraic logic calculator. (Do not look for a calculator with this exact display as most have additional keys like $\boxed{\%}$ and $\boxed{\sqrt{\ }}$ that are useful but not necessary to this discussion.) Some keys are marked with common abbreviations that we will continue to use in this text: CLR - clear; STO - store; RCL - recall; CHS - change sign. This last key is sometimes marked $\boxed{+/-}$ instead.

The logic of this machine is called algebraic but it doesn't follow the order rules of algebra you learned in Section 1.1. Calculations are fed into the machine much as you would type them on a typewriter (without spacing). Thus the multiplication 23 × 56 would be keyed

$$\boxed{2}\ \boxed{3}\ \boxed{\times}\ \boxed{5}\ \boxed{6}\ \boxed{=}$$

An instant after the $\boxed{=}$ is pressed the calculator displays the product

1288 *

Chains of operations may also be keyed directly into the calculator under certain conditions.

EXAMPLE 1.2 - 1. Calculate $21 \times 32 \times 61 \div 24$

Solution: Key:

$$\boxed{2}\ \boxed{1}\ \boxed{\times}\ \boxed{3}\ \boxed{2}\ \boxed{\times}\ \boxed{6}\ \boxed{1}\ \boxed{\div}\ \boxed{2}\ \boxed{4}\ \boxed{=}$$

Final Display: 1708.

EXAMPLE 1.2 - 2. Calculate $3.1 - 5.7 + 4.6$

Solution: Key:

$$\boxed{3}\ \boxed{\cdot}\ \boxed{1}\ \boxed{-}\ \boxed{5}\ \boxed{-}\ \boxed{7}\ \boxed{+}\ \boxed{4}\ \boxed{\cdot}\ \boxed{6}\ \boxed{=}$$

---

* We will not attempt to replicate calculator or computer displays in this text. Machines differ widely; most, however display numbers by lighting, filling with liquid, or printing some or all of seven small bars 8. These may be seen by looking closely. You might like to determine how many different displays could be made with these seven bars.

Display: 2

It is both interesting and useful to note that intermediate results are displayed on the calculator at various points in these chains of operations. In Example 1.2-1, for instance, when the second $\boxed{\times}$ is keyed in the calculation the display becomes

672

which is the product 21×32, the first two factors. Similarly when the $\boxed{\div}$ is keyed the display changes to

40992

the result of the calculation to this point $(21 \times 32 \times 61 = 40992)$. In exactly the same way in Example 1.2-2 the intermediate result

-2.6

is displayed when the $\boxed{+}$ is keyed.

It should be clear that algebraic logic is fine for chained computations that process left to right. But we saw in Section 1.1 that many computations do not have this simple order. Such computations lead to problems. To detect these problems the user must be alert; to solve them ingenuity must be exercised. The user must supply the one thing the calculator cannot: thinking! (This last sentence will, in fact, be a central message in all that follows.)

Consider the calculation

EXAMPLE 1.2-3. $\dfrac{2 \times 3}{5 \times 4}$

We know that the answer to this calculation is 3/10 or .3, and we would expect the calculator to display 0.3. You might attempt to carry this out by the following sequence.

$$\boxed{2}\ \boxed{\times}\ \boxed{3}\ \boxed{\div}\ \boxed{5}\ \boxed{\times}\ \boxed{4}\ \boxed{=}$$

The result of this sequence is 4.8, the wrong answer! Can you see what is incorrect in the calculation? The error is identified if the fraction is represented differently:

$$\frac{2\times 3}{5\times 4} = 2\times 3\times\frac{1}{5\times 4} = 2\times 3\times\frac{1}{5}\times\frac{1}{4} = 2\times 3\div 5\div 4$$

Thus, in general, each factor of the whole denominator is a divisor. This is a useful calculating technique to remember (but beware of applying it when the denominator is a sum. See example 1.2-4.)

A corrected calculation is

$$\boxed{2}\ \boxed{\times}\ \boxed{3}\ \boxed{\div}\ \boxed{5}\ \boxed{\div}\ \boxed{4}\ \boxed{=}$$

giving the correct result

0.3

A more difficult problem is presented by a calculation like:

EXAMPLE 1.2-4.

$$\frac{49+38}{85+96}$$

This time we have no direct solution technique. Several alternatives are available:

Solution (1) Calculate 49 + 38. Record the answer 87 on a scratch pad. Calculate 85 + 96. Record this answer, 181. Calculate 87 ÷ 181. This quotient, 0.4807* is the answer to the exercise.

There is nothing wrong with the solution shown here, but such a solution does not use the full power of the calculator. It is more than a matter of elegance not to have to write down such intermediate answers. Time may be lost and additional opportunities for error are accumulated as you copy and reenter numbers. Use of calculator storage (or memory) provides an alternative.

Solution (2)

$\boxed{8}\ \boxed{5}\ \boxed{+}\ \boxed{9}\ \boxed{6}\ \boxed{=}$ Calculate the denominator, 181.

$\boxed{\text{STO}}$ Store this number in calculator memory.

$\boxed{4}\ \boxed{9}\ \boxed{+}\ \boxed{3}\ \boxed{8}\ \boxed{=}$ Calculate the numerator, 87.

$\boxed{\div}\ \boxed{\text{RCL}}$ $\boxed{\text{RCL}}$ brings back the denominator 181 from memory.

$\boxed{=}$ The quotient 0.4807 is displayed.

---

* Results in this text will be given for 4-digit decimal rounding displays.

Exercise Set 1.2

Some of these exercises call for a calculator with simple algebraic logic with memory. (If your calculator also has parentheses keys, do not use them.)

1) Name four other keys that could replace $\boxed{=}$ in the calculation of Examples 1 and 2 to give the same answer. State a reason why you would not use these substitute keys if you were carrying out a series of calculations. (Try calculating $2 \times 3 \times$, followed without clearing by calculating $3 + 4$.)

2) One step in Solution 1.2-4(2) of the text may be eliminated. Examine the calculation carefully in order to find the extra step. Check your more elegant solution on a calculator.

3) In Solution 1.2-4(2) we calculated the denominator first. Try calculating the numerator first. What happens? (Some more sophisticated calculators have a key that switches the contents of the store and the display registers to avoid this kind of trap.)

4 - 10 Calculate, keeping intermediate record keeping to a minimum. Note which exercises require such records. RECALL THE ORDER RULES FROM SECTION 1.1.

4) $237 \times 42.5 + 38.46$

5) $39.42 + 861.7 \times 6.03$

6) $23.7 \div .06 \times 13.2$

7) $(78.35 + 91.46)(14.08 - 27.61)$

8) 2.8^3 Try to find an elegant way to calculate this.

9) $\frac{37.48 - 16.89}{64.32}$ 10) $\frac{64.32}{37.48 - 16.89}$

11) How should the answers to exercises 9 and 10 be related? Check this by calculation.

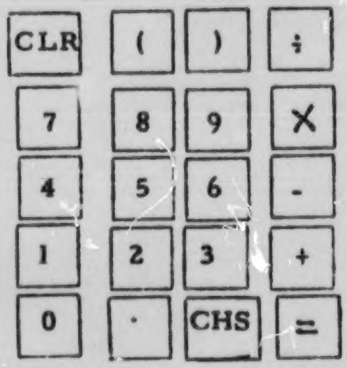
12 - 16 Calculate. Note intermediate records.

12) $\frac{239.5 - 67.34}{(74.2)(86.3)}$ 13) $\frac{(74.2)(86.3)}{239.5 - 67.34}$

14) (37.6 - 18.4)(15.2 - 83.1)(64.2 + 73.8) **Beware:** Some algebraic calculators allow the user only to add to or subtract from memory. If you are using one of those calculators, be sure to clear memory before storing a second number.

15) $\frac{(37.6 - 18.4)(15.2 - 83.1)}{64.2 + 73.8}$ 16) $\frac{4231(16.8 - 23.4)}{(83 - 1.3752)\,62.43}$

1.3 Calculator Logic: Algebraic with Parentheses*



The simple addition of parentheses to the algebraic keyboard simplifies much computation. The figure displays a keyboard for a calculator operating with this logic which we will call AP. Almost all such calculators have additional features like STO, RCL, √x, % and 1/x, but we restrict our discussion to the ones shown.

A quick comparison of this keyboard with the keyboard of Section 1.2 shows that only two keys are different: STO and RCL are replaced by (and). Surprisingly this minor modification makes the keying of complex calculations much simpler.

The main point to remember: Parentheses on the calculator play the same role of grouping computations that they do in algebra. There is, however, a difference in usage. The algebraic expression

$$a \left\{ b + (cd - e) \right\}$$

would appear as the calculator sequence

$$a \times (b + (c \times d - e)) =$$

---

[*] In sections 1.3 through 1.5 we will refer to algebraic logic with parentheses as AP in order to differentiate it from the algebraic memory logic of section 1.2.

Thus braces, brackets and other grouping symbols are all represented by the same symbols, parentheses.

EXAMPLE 1.3-1. $\dfrac{49 + 38}{85 + 96}$

(Recall that this was Example 1.2-4 of the last section.)

Solution:

$\boxed{4}\ \boxed{9}\ \boxed{+}\ \boxed{3}\ \boxed{8}$	Calculate the numerator
$\boxed{\div}$	Divided by. . . (numerator displayed: 87)
$\boxed{(}$	. . . the quantity . . . (signals a calculation to be done out of sequence)
$\boxed{8}\ \boxed{5}\ \boxed{+}\ \boxed{9}\ \boxed{6}$	Calculate the denominator
$\boxed{)}$	Completes the calculation in parentheses and displays it (181)
$\boxed{=}$	Displays the quotient of 87 - 181, 0.4807

Notice the effect of the right parenthesis, $\boxed{)}$:

(1) It plays the role of the $\boxed{=}$ key for the calculation since the most recent left parenthesis $\boxed{(}$ and displays the result.

(2) It "backs up" the calculation to where the left parenthesis $\boxed{(}$ was keyed. Thus the calculator acts as though you had just entered the calculated value of what is in parentheses.

WARNING: Parentheses do NOT represent multiplication!

The next example will show this.

EXAMPLE 1.3-2 -4.9 (3.7 - 8.9)

Solution:

$\boxed{4}\ \boxed{\cdot}\ \boxed{9}\ \boxed{\text{CHS}}$	Enter the multiplier, -4.9
$\boxed{\times}$	Multiplied by . . . (since the parentheses do not carry this meaning)
$\boxed{(}\ \boxed{3}\ \boxed{\cdot}\ \boxed{7}\ \boxed{-}$ $\boxed{8}\ \boxed{\cdot}\ \boxed{9}\ \boxed{)}$	Calculates the value of the expression in parentheses (-5.2)
$\boxed{=}$	Displays the product of -4.9 and -5.2, the answer 25.48

A modified algebraic logic that is closer to the rules of section 1.1 is called AH or AOS logic. Calculators like the TI-58 employ this logic. These letters represent the words Hierarchical Algebraic and Algebraic Operating System. With AH logic calculators the calculation

$$3 + 5 \times 7$$

could be keyed left to right without parentheses.

$$\boxed{3}\ \boxed{+}\ \boxed{5}\ \boxed{\times}\ \boxed{7}\ \boxed{=}$$

The calculator "remembers" when the $\boxed{\times}$ is pressed that multiplication takes precedence over addition.

AH calculators also require either memory or parentheses to process exercises like Example 1.3-1. On an AH calculator with parentheses the calculation would be:

$$\boxed{(}\ \boxed{4}\ \boxed{9}\ \boxed{+}\ \boxed{3}\ \boxed{8}\ \boxed{)}\ \boxed{\div}\ \boxed{(}\ \boxed{8}\ \boxed{5}\ \boxed{+}\ \boxed{9}\ \boxed{6}\ \boxed{)}\ \boxed{=}$$

Without the first pair of parentheses, the calculation would be for:

$$49 + \frac{38}{85 + 96}$$

Without either pair it would be

$$49 + \frac{38}{85} + 96$$

All AH calculators have both memory and parentheses as calculator functions.

Exercise Set 1.3

1 - 10 Rewrite each of the following expressions, for AP logic:

(a) removing parentheses that will not change the value algebraically

(b) removing parentheses that will not change the value in calculator computation

1) $3 + (5 - 7)$

2) $20 \times (10 \div 5)$

3) $\frac{(2 \times 7)}{(31 - 14)}$

4) $20 \div (10 \times 5)$

5) $(8 + 7)(3 + 5)$

6) $(27.3 + 41.7)3.6$

7) $27.3 + (41.7 \times 3.6)$

8) $(41.7 \times 3.6) + 27.3$

9) $41.7 \times (3.6 + 27.3)$

10) $\frac{(28 \times 3) + 8}{((26 + 7) \times 4)}$

11 - 14 Compute with an AP calculator:

11) $37.8 + (.06 \times 37.8)$

12) $1.06 \times 37\ 8$

13) $(2.8 + 4.5)^2(16 + 39.23)^2$

14) $\frac{26.4}{0631 - .1256}$

15 - 16 Calculate by algebraic-memory, by AP and by AH to compare procedures:

15) $\frac{327.84}{264 - 189}$ 16) $(48.3 + 27.9)(79.4 - 43.7)(67.1 - 4)$

17) In the song "The Twelve Days of Christmas", the lyrics begin:

"On the first day of Christmas
My true love gave to me
A partridge in a pear tree"

On the second day are given:

Two turtle doves and a partridge

On the third day:

Three French hens, two turtle doves and a partridge.

So it goes through twelve days until on the twelfth, for example, she receives:

"Twelve lords aleaping, eleven ladies waiting, ten...
[all the way down to] . . . a partridge in a pear tree."

Now suppose that on Christmas day the lovers break up and the gifts are returned one each day. For example, on the day after Christmas one of the partridges might be returned, the next day another, the following day another, the following day a French hen, and so on. When will all the gifts have been returned?

1.4 Calculator Logic: Reverse Polish Notation

The letters RPN represent Reverse Polish Notation, the country designated because the Polish logician J. Łukasiewicz developed the system. RPN is, in fact, often called Łukasiewicz logic. The reason for the R (Reverse) is that in this notation operation symbols are applied in order that is the reverse of what we learn in arithmetic and algebra. Thus

$$3 + 4 \quad \text{in RPN} \quad \text{is} \quad 3 \quad 4 \quad +$$

Think about that notation for a minute. What would happen if you keyed into any calculator:

$$\boxed{3}\ \boxed{4}\ \boxed{+}\ ?$$

It would record the number 34. Because of this problem an additional key appears on RPN calculator keyboards, the $\boxed{\text{ENTER}\uparrow}$ key.* Thus 3 + 4 is keyed:

$$\boxed{3}\ \boxed{\text{ENTER}}\ \boxed{4}\ \boxed{+}$$

On many RPN keyboards the ENTER key is larger because it is used so often.

Why would anyone want to change things around like that? It turns out that there are good reasons for doing so. If you examine Algebraic and RPN keyboards, you will see that the RPN ENTER replaces three algebraic keys:

---

* We will soon see that the arrow (↑) on this key has a role to play as well.

RPN	Algebraic or AH
$\boxed{\text{ENTER}}$	$\boxed{=}$ $\boxed{(}$ $\boxed{)}$*

We will now explore how this works.

All calculators must retain numbers and operations in memory during calculations. If this were not true, the calculator would "forget" the 3 when you sought to add 4 to it in the calculation 3 + 4. To accomplish this RPN calculators have what is called a stack.**

The calculator display is the "bottom" register of the stack. "Above" it are additional registers. Here is the four register stack found on a typical RPN programmable calculator:

REGISTER NAME		
T	0	
Z	0	
Y	0	
X	0	DISPLAY

The stack registers are arbitrarily named X, (the display register), Y, Z, and T, as shown.

As a number is entered in the stack it pushes other numbers up. When an operation is performed the stack (usually) moves down.

---

[*] RPN logic with an n-register stack also replaces n-2 algebraic storage registers.

[**] In fact all calculators have similar stacks. On algebraic logic calculators, for example, the $\boxed{(}$ key or even the $\boxed{+}$ key activate a stack. Because the stack plays a greater role in RPN, it is considered here in more detail.

EXAMPLE 1.4-1 Add 23 and 41

KEYS		STACK	
2 3	T	0	
	Z	0	
	Y	0	
	X	23	DISPLAY

Step 1. When 23 is keyed, it enters the X-register in the stack.

KEYS		STACK	
	T	0	
ENTER	Z	0	
	Y	23	
	X	23	DISPLAY

Step 2. When ENTER is keyed, the X-register is copied into the Y-register. (Y and Z registers also move up one level.)

KEYS		STACK	
	T	0	
4 1	Z	0	
	Y	23	
	X	41	DISPLAY

Step 3. When 41 is keyed, it REPLACES the contents of the X-register.

KEYS		STACK	
	T	0	
+	Z	0	
	Y	0	
	X	64	DISPLAY

Step 4. When + is keyed, it adds the X and Y registers. (T and Z registers also move down one level.)

The power of the ENTER key and the stack will be shown through a second example, a type of calculation that was a problem for us in AH and algebraic-memory logics.

EXAMPLE 1.4-2 Calculate $\dfrac{36.2}{25.8 - 28.3}$

KEYS **STACK**

$\boxed{3}\ \boxed{6}\ \boxed{\cdot}\ \boxed{2}$

T	0	
Z	0	
Y	0	
X	36.2	DISPLAY

Step 1. Key in 36.2. It appears in the X-register display.

$\boxed{\text{ENTER}}$

T	0	
Z	0	
Y	36.2	
X	36.2	DISPLAY

Step 2. ENTER copies X into Y.

$\boxed{2}\ \boxed{5}\ \boxed{\cdot}\ \boxed{8}$

T	0	
Z	0	
Y	36.2	
X	25.8	DISPLAY

Step 3. The 25.8 replaces 36.2 in the X-register display.

$\boxed{\text{ENTER}}$

T	0	
Z	36.2	
Y	25.8	
X	25.8	DISPLAY

Step 4. ENTER copies X and moves Y to Z. This second ENTER key allows us to calculate the denominator separately.

KEY **STACK**

$\boxed{2}\ \boxed{8}\ \boxed{\cdot}\ \boxed{3}$

T	0	
Z	36.2	
Y	25.8	
X	28.3	DISPLAY

Step 5. The 28.3 replaces 25.8 in the X-register display. Now all numbers are in the stack.

$\boxed{-}$

T	0	
Z	0	
Y	36.2	
X	-2.5	DISPLAY

Step 6. The contents of the X-register is subtracted from the Y-register. Z moves down to Y. The X-register now displays -2.5 = 25.8 - 28.3, the value of the denominator of the fraction being computed.

$$\frac{36.2}{25.8 - 28.3} = \frac{36.2}{-2.5}$$

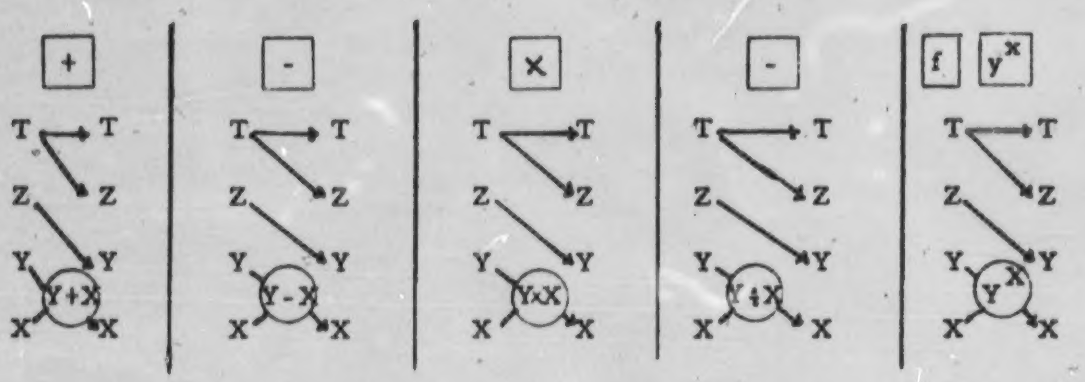
$\boxed{\div}$

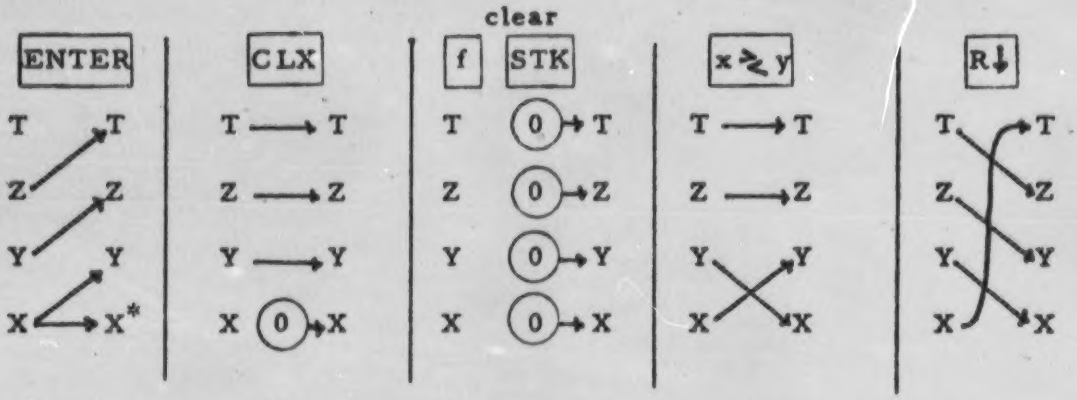
T	0	
Z	0	
Y	0	
X	-14.48	DISPLAY

Step 7. The Y-register is divided by the X-register and the answer displayed.

$$\frac{36.2}{25.8 - 28.3} = -14.48$$

The following diagrams will show how the registers in a 4 register stack change when various keys are depressed.





In RPN logic, use the ENTER key (for binary operations)

(1) after the first number in a calculation

(2) after the first number in a sub-calculation (the denominator of a fraction or any other calculation that would be placed in parentheses.)

A final calculator logic which we comment on only briefly is called Arithmetic logic. Arithmetic logic is like RPN for addition and subtraction and like algebraic logic for multiplication and division. The easiest

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[*] Recall that if a number is keyed next it will replace this.

way to identify Arithmetic logic calculators is by the combined function keys $\boxed{{}^{+}/_{=}}$ and $\boxed{{}^{-}/_{=}}$.

Many business calculators operate with Arithmetic logic. We will not refer to this logic again in this text.

Exercise Set 1.4

1 - 12 In each of the following exercises, the stack is shown as it was before the key is depressed. Show what the stack will be after the given key is depressed.

1) $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$ $\boxed{+}$

2) $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$ $\boxed{\div}$

3) $\begin{array}{|c|}\hline 8\\\hline 0\\\hline 2\\\hline 2\\\hline\end{array}$ $\boxed{\times}$

4) $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$ $\boxed{\text{ENTER}}$

5) $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$ $\boxed{5}$ BE CAREFUL!

6) $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$ $\boxed{\text{CLX}}$

7) $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$ $\boxed{\text{f}}\overset{\text{clear}}{\boxed{\text{STK}}}$

8) $\begin{array}{|c|}\hline 1\\\hline 1\\\hline 1\\\hline 1\\\hline\end{array}$ $\boxed{\text{CHS}}$

9) $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$ $\boxed{-}$

10) $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 0\\\hline\end{array}$ $\boxed{\div}$ WHAT DO YOU THINK ?

11) $\begin{array}{|c|}\hline 0\\\hline 0\\\hline 0\\\hline 35\\\hline\end{array}$ $\boxed{\text{ENTER}}$

12) $\begin{array}{|c|}\hline 35\\\hline 0\\\hline 0\\\hline 0\\\hline\end{array}$ $\boxed{\text{ENTER}}$

13 - 18. In the following exercises, show what the stack will be after each key is depressed.

13) $\boxed{\text{CLR}}$ $\boxed{5}$ $\boxed{\text{ENTER}}$ $\boxed{3}$ $\boxed{+}$

14) $\boxed{\text{CLR}}$ $\boxed{5}$ $\boxed{\text{ENTER}}$ $\boxed{\times}$

15) $\boxed{\text{CLR}}$ $\boxed{5}$ $\boxed{\times}$

16) $\boxed{\text{CLR}}$ $\boxed{3}$ $\boxed{\text{ENTER}}$ $\boxed{\text{ENTER}}$ $\boxed{+}$ $\boxed{\times}$

17) $\boxed{\text{CLR}}$ $\boxed{2}$ $\boxed{3}$ $\boxed{\text{ENTER}}$ $\boxed{5}$ $\boxed{\div}$

18) $\boxed{\text{CLR}}$ $\boxed{5}$ $\boxed{\text{ENTER}}$ $\boxed{4}$ $\boxed{\text{ENTER}}$ $\boxed{+}$ $\boxed{\div}$

19) Express in algebraic form the calculation carried out in exercises 14 - 18. (For example, exercise 13 is $5 + 3 = 8$.)

20 - 26 Give the RPN keystrokes for the following computations. Then calculate.

20) $(2 + 3)4$

21) $\frac{2}{3} + 4$

22) $4 + \frac{2}{3}$ (Hint: $\boxed{4}$ $\boxed{\text{ENTER}}$ $\boxed{2}$ $\boxed{\text{ENTER}}$ $\boxed{3}$ $\boxed{}$ $\boxed{}$)

23) $(2 + 3)(4 + 5)$

24) $\frac{2}{3} + \frac{4}{5}$

25) $\frac{2 + 3}{4 + 5}$

26) $(2 + 3)(4 + 5)(6 + 7)$

27) Show a second way to calculate exercise 26.

28) Give a keystroke sequence that will fill the stack in the following way

T	6	
Z	7	
Y	8	
X	8	DISPLAY

29) Recalculate exercises 16 and 18 with the stack at the beginning of the calculation in the form of exercise 28 and omitting the $\boxed{\text{CLR}}$ key. This exercise should show you that IT IS NOT NECESSARY TO CLEAR THE STACK IN ORDER TO CARRY OUT MOST CALCULATIONS.

30) Use an RPN calculator to compute the answer to exercise 17 of Section 1.3 (on page 1.3 - 5).

1.5 Other Calculating Keys

We have studied differences between algebraic-memory, AH and RPN logics. Most of these differences apply to binary operations, that is operations that "combine" two elements into one. Addition, subtraction, multiplication and division are the common binary operations of arithmetic. We met in section 1.4 one other, y^x, which we will consider shortly.

The following operations are unary operations, that is operations that need only one element to process.

$\sqrt{x}$
x^2
$1/x$
10^x
sine
cosine
tangent
CHS
INT
FRACT
ABS

(We will introduce other unary operations such as log x, ℓnx, and e^x, later.)

All calculators process unary operations by RPN! The x-value is keyed into the calculator and the function key is pressed.

EXAMPLE 1.5-1 Calculate $\sqrt{51}$

Keystroke sequence $\boxed{5}\ \boxed{1}\ \boxed{\sqrt{x}}$

Answer: 7.1414

EXAMPLE 1.5-2 Calculate sin 30°

Keystroke sequence* $\boxed{3}\ \boxed{0}\ \boxed{\text{SIN}}$

Answer: .5

EXAMPLE 1.5-3 Find the reciprocal of 10^2

Keyboard sequence 1: $\boxed{1}\ \boxed{0}\ \boxed{x^2}\ \boxed{1/x}$

Keyboard sequence 2: $\boxed{2}\ \boxed{10^x}\ \boxed{1/x}$

Answer: .01

In all cases these function keys operate on the number in the display or X register. Note that it is not necessary to depress the ENTER key on an RPN-calculator before using them. For any unary function f the stack diagram is:

$$\boxed{f}$$

$$\begin{array}{lcl} T & \longrightarrow & T \\ Z & \longrightarrow & Z \\ Y & \longrightarrow & Y \\ X & f(x) \rightarrow & X \end{array}$$

The unary functions INT, FRACT and ABS will be considered in the exercises.

One important function that does differ between AH and RPN calculators is exponentiation (raising to a power). This is a binary operation because

$$p^q$$

requires the two input elements p and q.

---

* Most calculators assume input to trigonometric functions to be in degrees. We will also unless otherwise mentioned. See Chapter 7.

EXAMPLE 1.5-4 Compute 7^4

AH keystroke sequence: $\boxed{7}\ \boxed{y^x}\ \boxed{4}\ \boxed{=}$

RPN keystroke sequence: $\boxed{7}\ \boxed{\text{ENTER}}\ \boxed{4}\ \boxed{y^x}$

Answer: 2401

When using this $\boxed{y^x}$ key, you will meet for the first time the fact that calculators sometimes produce only approximate answers. In the calculation of 7^4, for example, a calculator might display the answer 2400.9993. Now we know that 7^4 is an integer and we can find it exactly by multiplying $7 \times 7 \times 7 \times 7$ to get 2401. The error (of -.0007 in this case) is introduced by the logarithmic processing used by the calculator $\boxed{y^x}$ key. We will study this later. For now it is usually enough to round off such answers to the nearest integer.

Exercise Set 1.5

1 - 17 Without using a calculator give the display produced by the following keystroke sequences. Check your results by calculator.

1) $\boxed{5}\ \boxed{x^2}$ 2) $\boxed{9}\ \boxed{\sqrt{x}}$ 3) $\boxed{4}\ \boxed{1/x}$

4) $\boxed{3}\ \boxed{10^x}$ 5) $\boxed{8}\ \boxed{\text{CHS}}$ 6) $\boxed{0}\ \boxed{1/x}$

7)* $\boxed{5}\ \boxed{\text{ENTER}}\ \boxed{x^2}$ What does the stack look like after this sequence?

8) $\boxed{5}\ \boxed{\text{ENTER}}\ \boxed{x^2}\ \boxed{+}$ 9) $\boxed{4}\ \boxed{10^x}\ \boxed{\sqrt{x}}$

---

[*] Note that the keys will tell which logic is used. An AH calculator has no ENTER key.

10) $\boxed{7}\ \boxed{\text{CHS}}\ \boxed{x^2}$ 11) $\boxed{2}\ \boxed{x^2}\ \boxed{1/x}$ 12) $\boxed{2}\ \boxed{1/x}\ \boxed{x^2}$

13) $\boxed{9}\ \boxed{1/x}\ \boxed{x}$ 14) $\boxed{5}\ \boxed{\text{ENTER}}\ \boxed{1/x}\ \boxed{x}$

15) $\boxed{1}\ \boxed{10^x}$ 16) $\boxed{(}\ \boxed{2}\ \boxed{+}\ \boxed{3}\ \boxed{)}\ \boxed{10^n}$

17) $\boxed{9}\ \boxed{+}\ \boxed{7}\ \boxed{=}\ \boxed{\sqrt{x}}$

18 - 20 By applying the functions to various values, determine what the following keys do. Be sure to include values like 7.65, -3, -3.72.

18) INT 19) FRACT 20) ABS

21 - 26 Calculate each of the following. Check your result against the answer given.

EXAMPLE: $5 + \sqrt{7}$

AH keystroke*: $\boxed{5}\ \boxed{+}\ \boxed{7}\ \boxed{\sqrt{x}}\ \boxed{=}$

RPN keystroke: $\boxed{5}\ \boxed{\text{ENTER}}\ \boxed{7}\ \boxed{\sqrt{x}}\ \boxed{+}$

Answer: 7.6458

21) 8^5 Ans. 32768 22) 1.23^3 Ans. 1.8609

23) $\frac{1}{16} + \frac{1}{7}$ Ans. 0.2054 24) $\frac{1}{16 + 7}$ Ans. 0.0435

25) $10^5 - 5^7$ Ans. 218745 (On some algebraic calculators you may find it necessary to use parentheses around 5^7.)

26 - 28 Now try your hand at the following monsters:

26) $\frac{\sqrt{35}\,\sin 45^\circ}{34^3}$ Note: the numerator is a product. Ans. 0.0001

27) INT $(10^{\tan 60^\circ})$ Ans. 53

---

* Some AH calculators will not accept this calculation. On them it must be reordered to $\sqrt{7} + 5$, or it must be calculated as $5 + (\sqrt{7})$.

28) $\dfrac{\sqrt{3.7} + \cos 10^{\circ}}{.13^{3} - \frac{1}{27}}$ Ans. -83.4771

The following two exercises provide useful short-cuts for computation:

29) Sometimes the wrong number appears in the display. For example, when you wish to calculate $a - b$, b may already be displayed. How could you complete the calculation without starting all over?

20) How can you calculate $\frac{a}{b}$ starting with b in the display?

1.6 Problem Solving with a Calculator

With the power your calculator gives you, you may now attack with confidence and solve some complicated problems. You will need paper and pencil only to record notes and answers. As you will see, however, the calculator does not substitute for thinking. You are still in charge. You will need to

- organize calculations so that you can carry them out on your calculator

and if your problem is one related to measurement

- determine units for the answer and
- determine accuracy

In this section we will not deal with the latter two important questions. We will continue to report answers to four digit (rounded) accuracy*.

EXAMPLE 1.6-1 A simplified formula for artillery range is

$$R = \frac{Vo^2 \sin A \cos A}{9.8}$$

Find the numerical value (without units) of R when

$$Vo = 31 \text{ and } A = 30°.$$

---

* Some calculators truncate answers rather than round answers. Truncated means that the rest of the answer is cut off. Thus 683.29587 truncated to six digits is 683.295, the .00087 merely dropped. This is often called "rounding down". You should test your calculator to see how it rounds Use quotients like 2/3, 5/33, and 50/33.

SOLUTION: substituting

$$R = \frac{31^2 \sin 30^\circ \cos 30^\circ}{9.8}$$

Calculation yields 42.4618

Such a calculation is important but straightforward. Others require an experimental approach.

EXAMPLE 1.6-2 In EXAMPLE 1 we might wonder what angle A makes R largest. (What angle of elevation yields longest range?)

SOLUTION. We need only consider the product

sin A cos A (Why?)

Trying values yields

A	sin A cos A
30°	0.4330
40°	0.4924
50°	0.4924
60°	0.4330

This suggests trying A = 45° (Why?)

45°	.5

Trying other values suggests that this is the best we can get in the range 0° to 90°.

Often it simplifies computation to use storage capacity of your calculator to evaluate expressions in which letters appear more than once. In the following example, we assume a calculator that has at least two storage registers R1 and R2. To store 5 in R1 and 6.3 in R2 the following keystroke sequence could be used:

$$\boxed{5}\ \boxed{\text{STO}}\ \boxed{1}\ \boxed{6}\ \boxed{\cdot}\ \boxed{3}\ \boxed{\text{STO}}\ \boxed{2}$$

This sequence of keys is appropriate for most algebraic or RPN calculators.*

To recall the number in R1, you need only press

$$\boxed{\text{RCL}}\ \boxed{1}$$

and in this case the 5 will reappear in the display.

EXAMPLE 1.6-3. Evaluate $x^3 + 3x^2y + 3xy^2 + y^3$ for $x = 3.7$ and $y = 8.6$.

SOLUTION. If you attack this problem directly you will be keying 3.7 and 8.6 each several times. You can save some of these keystrokes by first storing x and y. Follow the program for the kind of calculator you use.

---

* On some calculators each register may have a two-digit designation. In that case to store 5 in R01 would be keyed

$$\boxed{5}\ \boxed{\text{STO}}\ \boxed{0}\ \boxed{1}\,.$$

$\boxed{3}\ \boxed{\cdot}\ \boxed{7}\ \boxed{\text{STO}}\ \boxed{1}$ * $\boxed{8}\ \boxed{\cdot}\ \boxed{6}\ \boxed{\text{STO}}\ \boxed{2}$

or

AH	representing	RPN
$\boxed{\text{RCL}}\ \boxed{1}\ \boxed{y^x}$ ** $\boxed{3}$ $\boxed{+}$	x^3	$\boxed{\text{RCL}}\ \boxed{1}$ * $\boxed{3}\ \boxed{y^x}$ **
$\boxed{(}\ \boxed{\text{RCL}}\ \boxed{1}\ \boxed{y^x}\ \boxed{2}$ $\boxed{\text{X}}\ \boxed{\text{RCL}}\ \boxed{2}\ \boxed{\text{X}}\ \boxed{3}$ $\boxed{)}$ $\boxed{+}$	$3x^2y$	$\boxed{\text{RCL}}\ \boxed{1}$ * $\boxed{2}\ \boxed{y^x}\ \boxed{\text{RCL}}$ $\boxed{2}\ \boxed{\times}\ \boxed{3}\ \boxed{\times}$ $\boxed{+}$
$\boxed{(}\ \boxed{\text{RCL}}\ \boxed{2}\ \boxed{y^x}\ \boxed{2}$ $\boxed{\text{X}}\ \boxed{\text{RCL}}\ \boxed{1}\ \boxed{\text{X}}\ \boxed{3}$ $\boxed{)}$ $\boxed{+}$	$3xy^2$	$\boxed{\text{RCL}}\ \boxed{2}$ * $\boxed{2}\ \boxed{y^x}\ \boxed{\text{RCL}}$ $\boxed{1}\ \boxed{\times}\ \boxed{3}\ \boxed{\times}$ $\boxed{+}$
$\boxed{(}\ \boxed{\text{RCL}}\ \boxed{2}\ \boxed{y^x}\ \boxed{3}$ $\boxed{)}$ $\boxed{=}$	y^3	$\boxed{\text{RCL}}\ \boxed{2}$ * $\boxed{3}\ \boxed{y^x}$ $\boxed{+}$

You should reach the value 1860.867

---

[*] At each of these points the $\boxed{\text{ENTER}}$ key is omitted because the unary operation key substitutes for it.

[**] Beware! Do not confuse the X and Y registers with the x and y in the polynomial. The $\boxed{y^x}$ key operates on numbers in the appropriate calculator registers.

In solving complex problems like these you will need to be very careful. Here are some suggestions which may help:

(1) Think through your computation before you start to key numbers into the calculator.

(2) Try to organize your computation in parts such as terms of a polynomial or the numerator and denominator of a fraction.

(3) If you feel you will be lost computing the answer to a complex problem in one series of keystrokes, take it part by part, recording partial answers. You may then combine these into a final solution.

(4) Sometimes (as you will see in the exercises) algebraic simplification of an expression to be evaluated will also simplify computation.

Exercise Set 1.6

1 - 4 Evaluate the formula $R = \frac{Vo^2 \sin A \cos A}{9.8}$ for R using the given values of Vo and A.

1) $Vo = 200$, $A = 40^\circ$

2) $Vo = 100$, $A = 40^\circ$

3) $Vo = 100$, $A = 50^\circ$

4) $Vo = 375$, $A = 90^\circ$

(You may wish to think about the results of exercises 1 - 4 as they relate to the physics of projectile range.)

5 - 9 Using one of the two conversion formulas for Celsius and Fahrenheit temperatures,

$$C = \frac{5}{9}(F - 32) \qquad F = \frac{9}{5}C + 32$$

answer the following:

5) $C = 100^\circ$, find F

6) Convert $32^\circ F$ to C.

7) Change $68^\circ F$ to C

8) Change $98.6^\circ F$* to C.

9) Find by experimenting when F and C are the same.

10) Now check your answer to exercise 9 by algebra. (Set F and C each equal to x in one of the two formulas and solve for x.)

11-14. A formula for triangle area that you will be able to derive later is

$$A = (s_1 \cdot s_3 \cdot \sin A_2) \div 2$$

Find A, given the following values:

11) $s_1 = 2$, $s_3 = 5$, $A_2 = 50^\circ$

12) $s_1 = 10$, $s_3 = 8$, $A_2 = 60^\circ$

13) $s_1 = 3.72$, $s_3 = 5.8$, $A_2 = 38^\circ$

14) $s_1 = 147.3$, $s_3 = 62.1$. $A_2 = 72^\circ$

15-19. If an object is h meters above the ground, the time, t, in seconds, that it takes to fall to the ground is given by the formula $t = \sqrt{\frac{2h}{9.8}}$

Find t when:

15) $h = 147.2$

16) $h = 3472.13$

17) $h = 1.23$

18) $h = 43.278$

19) Solve the formula for h and use your new formula to find h when $t = 10$.

20-23. In a right triangle whose legs are a and b and whose hypotenuse is c, you know that $c = \sqrt{a^2 + b^2}$.

---

* Average human body temperature.

Find c when:

20) $a = 5$ and $b = 4$

21) $a = 10.35$ and $b = 15.72$

22) $a = 10.3$ and $b = 11.7$

23) $a = 2.3$ and $b = 18.9$

24-30. Evaluate when $x = 3.7$ and $y = 8.6$. Store these values for x and y.

24) $x^2 + 2xy + y^2$

25) $(x + y)^2$

26) $x^2 + y^2$

27) $(x + y)^3$

28) What identity to your answers in exercises 24 and 25 support?

29) What do your answers in 24, 25 and 26 suggest?

30) What identity does your answer to exercise 27 and the answer to example 3 on pages 1.6-3 and 4 suggest?

1.7 Programming Functions: 1

In working the exercises of Section 1.6 you should have found the calculations repetitious. You were following similar routines over and over, with only the numbers different. In this section we will develop a short-cut to reduce such work.

EXAMPLE 1. Give a keystroke routine that will start with a given value of h and calculate t by the formula

$$t = \sqrt{\frac{2h}{9.8}}$$

SOLUTION

AH	RPN
key in h, then	key in h, then
$\boxed{\times}$ $\boxed{2}$	$\boxed{\text{ENTER}}$
$\boxed{\div}$ $\boxed{9}$ $\boxed{\cdot}$ $\boxed{8}$ $\boxed{=}$	$\boxed{2}$ $\boxed{\times}$
$\boxed{\sqrt{x}}$	$\boxed{9}$ $\boxed{\cdot}$ $\boxed{8}$ $\boxed{\div}$
	$\boxed{\sqrt{x}}$

Notice that once the keystrokes have been worked out it requires no knowledge of the function to follow them. With these instructions you could give your homework exercises to an elementary school aged sister or brother to calculate for you. For example, given the h value 10, they would key $\boxed{1}$ $\boxed{0}$ and then the keystrokes for your calculator, giving the resulting t value, 1.4286.

Still better, you can assign this routine to your programmable calculator. Here are the general steps you can use to accomplish this:

(1) Set your calculator or computer to record a program.

(2) Key into your calculator or computer the calculation steps (along with any instruction steps necessary to your particular calculator.)

(3) Set your calculator or computer back to calculating mode.

(4) If necessary reset your calculator or computer to the beginning of your program.

(5) Enter your given data.

(6) Run the program.

For additional exercises of the same type, you then merely repeat steps 4, 5, and 6.

Each of the many calculator or computer models operates differently so it is not possible to list all the special instructions required to carry out the six step routine we have just given. Because they suggest the kinds of special differences you will meet on calculators and computers, we offer three examples here. You should study them to see their form, but you should concentrate on the specific routines for the calculator or computer you will be using. Recall that we are programming the calculation

$$\sqrt{\frac{2h}{9.8}}$$

TI-58 (a typical AH programmable calculator)

1.	OFF - ON	This clears the calculator of previous programs.*
	LRN	LRN sets this calculator to record the program. The display is 000 00
2.	X 2	These are the calculation steps. (See the SOLUTION to EXAMPLE 1 on page 1.7 - 1.
	÷ 9 . 8	As you depress each key the calculator display will move to a new step number 001 00 up to 008 00.
	- $\sqrt{x}$	
	R/S RST	R/S is required to stop the program and display the result of the calculation. RST will return the program to 00.
3.	LRN	This key now returns the calculator to normal operation. The display is 0
4.	RST	RST sets the calculator to run from step 000.
5.	Key in h.	
6.	R/S	R/S then activates the program. When the calculator stops, the display will give the t value.

To find additional pairs (h, t), repeat steps 5 and 6. By inserting the RST after the R/S at the end of step 2, we don't need to repeat step 4 each time we run the program.

Note: On this calculator RST plays a different role within a program (as in step 2) and outside (as in step 4). Within a program RST returns the calculator to 00 and contines to run. Outside a program RST returns the calculator to 00 and stops there.

---

* Some calculators, usually with a C designation - as TI-58C - have continuous memory and must be cleared by other means. See the owner's manual.

HP-33E (a typical RPN programmable calculator)

1.	OFF - ON	This clears the calculator of previous programs.
	PRGM	PRGM sets the calculator to record your program steps. The display is 00.
2.	ENTER↑ 2 × 9 . 8 ÷ f* $\sqrt{x}$	These are the calculation steps. As you depress each key the display records the step number and the location (row - column) of the key(s) depressed. For example, after ENTER is pressed 01 31 is displayed: 01 is the step number, 31 the location (row 3, key 1) of ENTER on the keyboard. Note how the last two keys are merged into one step 08 14 02. This saves program steps.
3.	RUN	The calculator is now returned to normal operation, the display is 0.00.
4.	g RTN	The RTN key sets the program back to step 00.
5.	Key in h	
6.	R/S	This activates the program. On this calculator we did not have to key another R/S into the program because all unused program steps are pre-loaded with steps that return the program to step 00 and stop it there.

To find additional pairs (h, t) repeat steps 5 and 6. On this calculator it is not necessary to repeat step 4 because the calculator itself resets to step 00 at the end of a program run.

---

[*] On this calculator, as on many others, many keys have two or even three roles. Here the yellow f key assigns the second role $\sqrt{x}$ to the 2 key. The blue g key would have assigned x^2 to the same key.

TRS-80 (a typical computer programming in BASIC)

1.	OFF - ON	A prompt > is displayed.
2.	ENTER	This prepares your computer for further instructions. You must press ENTER at the beginning of your program and after each step.
3.	AUTO	This places you in automatic and numbers the steps in your program. It first prints 10, ready for your program.
4.	10 INPUT H 20 T = SQR (2*H/9.8) 30 PRINT T	These are the calculation steps. Each time you finish a line and press ENTER the computer goes to the next line and prints the next number.
5.	BREAK	This takes you out of automatic and puts you in run mode.
6.	Type in the value of H	
7.	Type RUN	This activates the program.

To find additional pairs (h, t) repeat steps 6 and 7.

You should familiarize yourself with the procedures for entering and running programs, but the more important task is developing programs. Here are some suggestions about how to do this:

(1) Remember that the program merely records what you would have done in a calculation that is not programmed.

(2) Think of your calculation as always starting from the value(s) that change in the computation (In the example this was h.)

(3) Key into the program the steps following (and not including) the step that keys your starting value (see suggestion 2) into the display. (On an RPN calculator don't forget ENTER when it is necessary.)

(4) Be sure, if your calculator or computer requires it, to complete your program with $\boxed{\text{R/S}}$ so your calculator or computer will stop to display the results.

Exercise Set 1.7

1 - 6. Key into your calculator or computer a program to find t, given h, by the formula

$$t = \sqrt{\frac{2h}{9.8}}$$

Then calculate t for the following h values

1) 15 2) 100

3) 1000 4) 10,000

5) 8840 (m in ht. of Mt. Everest) 6) 1609 (m in a mile)

7) Develop a keystroke sequence to change any Fahrenheit temperature into Celsius by the formula

$$C = \frac{5}{9}(F-32)$$

(Don't forget to start your calculation from F.)

8-12. Program the calculation of exercise 7 and use it to convert the following temperatures to Celsius:

8) 0° F 9) 90° F

10) 50° F

11) -40° F

12) By experimenting, find when F = 2C, that is when Fahrenheit temperature is twice Celsius temperature.

13) The sales tax in Erie County, New York is 7%. Develop a keystroke sequence that will calculate the amount of this sales tax. (Do not bother with rounding your answer.)

14-18. Program the calculation of exercise 13 and use it to determine sales tax on the following purchases:

14) $500

15) $45.32

16) $299.95

17) $2.79

18) By experimenting, find a purchase price that will give a sales tax of $1.00.

1.8 Programming Functions: 2

In section 1.7 you learned to program your calculator or computer so that it would carry out computation routines by a single keystroke. In that section you were restricted to single input-single output routines. Now in this section you will learn how to handle more than one input or output.

On a calculator the key to this problem and the key to press is:

$$\boxed{\text{R/S}}$$

This powerful key plays the following important roles:

1. When the calculator is in operating mode, it either starts a program if the calculator is idle or stops a running program.
2. When it is keyed into a program it stops the program either to receive information or to give information.

We will consider how this works by means of examples.

EXAMPLE 1.8 - 1. Develop and run a program to evaluate $\underline{c}$ for entered values of $\underline{a}$ and $\underline{b}$, in the formula

$$\underline{c} = \sqrt{\underline{a}^2 + \underline{b}^2}$$

SOLUTION 1.

by TI 58		by HP 33E
OFF - ON		OFF - ON
$\boxed{\text{LRN}}$	a would be keyed before the program started	PRGM
$\boxed{x^2}$ $\boxed{+}$		$\boxed{g}$ $\boxed{x^2}$
* $\boxed{\text{R/S}}$	Here the calculator is stopped to receive b.	* $\boxed{\text{R/S}}$
$\boxed{x^2}$		$\boxed{g}$ $\boxed{x^2}$
$\boxed{=}$		$\boxed{+}$
$\boxed{\sqrt{x}}$		$\boxed{f}$ $\boxed{\sqrt{x}}$
$\boxed{\text{R/S}}$		
$\boxed{\text{LRN}}$		RUN

Here is how these programs would be run for a = 5, b = 12.

$\boxed{\text{RST}}$	Resetting the program to 0	$\boxed{g}$ $\boxed{\text{RTN}}$
$\boxed{5}$ $\boxed{\text{R/S}}$	Enter 5 and start the program	$\boxed{5}$ $\boxed{\text{R/S}}$

The first part of the program runs until it reaches $\boxed{\text{R/S}}$ at the step marked * in the program. It stops with the display reading 25.

$\boxed{1}$ $\boxed{2}$ $\boxed{\text{R/S}}$	Enter 12 and restart	$\boxed{1}$ $\boxed{2}$ $\boxed{\text{R/S}}$

Either calculator will now complete its program and display the c value 13.

You will develop other ways of carrying out this kind of multiple input program in the exercises. We now consider a problem involving multiple output.

EXAMPLE 1.8-2 Develop a program that will calculate and display sales tax (at 7%) and then total cost for given purchase prices.

TI 58

OFF - ON* LRN	
STO 0 1	Stores purchase price
X • 0 7 =	Calculates sales tax
R/S	Stops to display tax
+ RCL 0 1 =	Adds on purchase price to give total cost.
R/S	
LRN	

Running the program for a $92 purchase

RST 9 2 R/S

The calculator runs to the first R/S, and stops there to display the sales tax $6.44.

R/S The calculator completes the program and displays the total cost $98.44.

---

* On this calculator there are other ways to clear programs and reset the program to 0, but we adopt this simple procedure. In fact, new programs may be keyed right "over" old ones for the new steps replace the old.

HP 33E

Storage could be used as in the TI-58 solution but instead we utilize the operating stack to solve this problem.

OFF - ON*, PRGM

ENTER ENTER	Now the purchase price is in Y, and Z registers.
. 0 7 X	(or 7 g %) Now sales tax is in X, purchase price in Y.
R/S	Stop to display sales tax
+	Adds sales tax and purchase price.
RUN	

Running the program for a $92 purchase

g RTN	Resets to 00
9 2 R/S	

Now the calculator displays the sales tax $6.44

R/S	The calculator completes the program and displays the total cost $98.44.

---

* On this calculator also there are other ways to clear programs and reset the program to 00, but we adopt this simple procedure. In fact, again new programs may be keyed right "over" old ones for the new steps replace the old.

TRS-80

```
ON, ENTER, AUTO              You are now ready to write
                             your program.

10   INPUT  P

20   T = .07 * P             Computes tax.

30   C = T + P               Adds tax to purchase.

40   PRINT  P, T, C          Prints purchase price, sales
                             tax and final cost.

BREAK
```

Running the program for a $92 purchase

```
92

RUN
```

The microprocessor prints

```
92,   6.44,   98.44
```

Exercise Set 1.8

1 - 4 Program EXAMPLE 1.8-1 into your calculator and use this program to find c for the following:

1) a = 23, b = 264

2) a = 45, b = 24

3) a = 45, b = 336

4) a = 7, b = 24

5 - 9 For a = 45, there are five other values of b that result in Pythagorean triples, that is results for a, b, and c all in integers.

Find the $\underline{b}$ and $\underline{c}$ that completes the ($\underline{a} = 45$) triple for $\underline{b}$ in each of the following ranges:

5) $25 \le b \le 30$

6) $60 \le b \le 65$

7) $105 \le b \le 110$

8) $195 \le b \le 200$

9) $1010 \le b \le 1015$

10 - 15 Program EXAMPLE 1.8-2 into your calculator or computer and use this program to find sales tax and total cost for the following purchase prices.[*]

10) $34.95

11) $1.67

12) $2995

13) $632.50

14) $99.95

15) $100

16) How could you modify the program of EXAMPLE 2 if sales tax went up to 8%? Clearly you can start over and reenter the entire program, but you may wish to experiment with calculator keys in LRN or PRGM mode to make the necessary key change. You will need to determine how the following keys work on your calculator:

---

[*] On calculators that display four decimal digits (like the HP 33E) you need to exercise care here. Such calculators probably do not round up but either round down (truncate) or round to the nearest value. Your best procedure is to reset such calculators to display more decimal digits. On the HP 33E, for example, to set three decimal places in the display press $\boxed{\text{f}}$ $\boxed{\text{FIX}}$ $\boxed{3}$.

$\boxed{\text{SST}}$	single step
$\boxed{\text{BST}}$	back step

and on the TI 58

$\boxed{2^{nd}}$ $\boxed{\text{INS}}$	insert
$\boxed{2^{nd}}$ $\boxed{\text{DEL}}$	delete

17) (for algebraic calculators or computers only) In EXAMPLE 1.8-2 you had to store the purchase price because it is lost when you calculate sales tax. Show how you can avoid storage by calculating total cost from the sales tax. (HINT: If purchase price is p, sales tax is .07p and total cost is 1.07p. Determine the number you must multiply .07p by to get 1.07p.)

18) Suppose you were a householder in an area where different communities in which you shopped charged different sales taxes. (This is fairly common near state or even county boundaries.) Develop a sales tax - total cost program so that you can enter list price and then sales tax rate to produce sales tax and total cost. (Hint: an easy way to do this is to use program storage.) Use your program to complete the following table:

	list price	tax rate	tax	cost
suit	$137.95	6%		
overcoat	84.50	8%		
shoes	31.45	7%		
hat	18.50	6%		
TOTALS	----	---		

1.9 Programming Algorithms 1.

In sections 1.2 to 1.8 we utilized particular calculators to solve problems. In those sections we introduced some ideas that are quite general and apply to programmable calculators and computers of many types. The method of entering and running a program to evaluate an often-repeated calculation is, for example, common to all programmable calculators and computers. At the same time many of the ideas were specific to the particular calculating device we used. In this category are, for example, the different operational systems - AH and RPN - the specific means of switching into program mode - PRGM or LRN - and the key for resetting the calculator to run through a program again - RST or RTN.

From now on we wish to provide more general instructions which will apply to any calculator or computer. It is then usually quite easy to translate the given procedure or algorithm into a keystroke routine for your particular calculator, microprocessor, or computer. We will work through an example to show how an algorithm is arrived at and then how it may be translated into specific routines. In the example and in subsequent work you will be led to develop algorithms.. Translating them into keystroke sequences will be done with the guidance of your teacher.

EXAMPLE 1.9.1 Find the real and imaginary coefficients of the product $(3 + 2i)(5 + 7i)$

SOLUTION: By standard algebraic techniques we have

$$(3 + 2i)(5 + 7i) = 15 + 31i + 14i^2$$

and since $i^2 = -1$

$= 15 + 31i - 14 = 1 + 31i$

Real coefficient: 1

Imaginary coefficient: 31

EXAMPLE 1.9-2 Find the real and imaginary coefficients of the product $(a + bi)(c + di)$

SOLUTION: $(a + bi)(c + di) = ac + adi + bci + bdi^2$

$= ac + (ad + bc)i - bd = (ac - bd) + (ad + bc)i$

Real coefficient: $ac - bd$

Imaginary coefficient: $ad + bc$

Notice how the second example generalizes the first. It also suggests an algorithm for calculating the real and imaginary coefficients of the product of any two complex numbers, $a + bi$ and $c + di$:

EXAMPLE 1.9-3 Multiply two complex numbers:

Given a, b, c, and d. Finding e and f where

$e + fi = (a + bi)(c + di)$:

Solution: Notice that each of the numbers a, b, c, and d is used twice:

first use of d

$e = ac - bd$

second use of d

$f = ad + bc$

Therefore we need to store these numbers in memory.

MULTIPLYING TWO COMPLEX NUMBERS

1. Remember a, b, c, and d. (In a computer, this could be done with LET commands, in a calculator with keys such as STO.)
2. $e \leftarrow ac - bd$; display the result*
3. $f \leftarrow ad + bc$; display the result
4. STOP

A good algorithm has certain features:

- variables are initialized (or introduced)
- calculations are made or decisions are reached, based on the values of the variables;
- all possibilities are accounted for
- information is displayed or printed
- the process has a way to stop.

A verbal algorithm is just an abbreviated statement of the steps we use to solve a problem. For us it will play a role between a complex problem and a keystroke sequence for our particular calculating device. For the algorithm MULTIPLYING TWO COMPLEX NUMBERS, consider now how the steps could be translated into program keystrokes.

---

* the arrow notation means "replace the value of ... with the value of ...". Thus, $e \leftarrow ac - bd$ means replace the value of e with the value of ac - bd.

1. Remember a, b, c, and d.

TI 58		HP 33E	TRS-80
STO 00	(Ro = a)[*]	STO 0	INPUT A, B, C, D
R/S		R/S	
STO 01	(R_1 = b)	STO 1	
R/S		R/S	
STO 02	(R_2 = c)	STO 2	
R/S		R/S	
STO 03	(R_3 = d)	STO 3	

2. Compute ac - bd; display the result.

RCL 00	RCL 0	R = A*C-B*D
×	RCL 2	PRINT R
RCL 02	×	
−	RCL 1	
RCL 01	RCL 3	
×	×	
RCL 03	−	
=	R/S	
R/S		

---

[*] We will use the notation R_n to denote storage register n.

3. Compute ad + bc; display the result.

TI-58	HP 33E	TRS-80
RCL 00	RCL 0	I = A*D + B*C
X	RCL 3	PRINT I
RCL 03	X	END
+	RCL 1	
RCL 01	RCL 2	
X	X	
RCL 02	+	
=		
R/S		

4. Stop

Last command in 3 (RST)*	Calculator automatically resets to 00 and stops	Last command in 3

Exercise Set 1.9

1 - 4 What do each of the following verbal algorithms calculate?

1) (1) Enter a, b

(2) Compute $\sqrt{a^2 + b^2}$; display the result

(3) Stop

2) (1) Enter F

(2) Compute $\frac{5}{9}(F-32)$; display the result

(3) Stop

---

[*] By including RST in your program, you will not have to key this before each run; however, you will still have to key RST before your first run.

3)
```
(1) Remember p
(2) $t \leftarrow .07p$; display t
(3) $s \leftarrow p + t$; display s
(4) stop
```

Note: We will use the notation $x \leftarrow$ to mean compute x from what is to the right of the arrow.

4)
```
(1) Remember a, b
(2) $c \leftarrow a^2 - b^2$; display c
(3) $d \leftarrow 2ab$; display d
(4) stop
```

Hint: How is $c + di$ related to $a + bi$?

5) What is usually the last step in a verbal algorithm? (We will see exceptions to this later.)

6 - 9 Often we wish to replace one variable by some function of that same variable. Thus we might use $x \leftarrow x + 1$ to mean "replace x by x + 1" or "make x one larger." Translate each of the following into statements beginning "Replace. . . ."

6) $x \leftarrow 2x$

7) $x \leftarrow 1/x$

8) $x \leftarrow 5$

9) $x \leftarrow x - 1$

10 - 12 Each of the following verbal algorithms lacks <u>one</u> of the features of a good algorithm (listed on page 1.9-3). Tell what that feature is and give a step to make the program complete.

10)
```
(1) Enter x
(2) $x \leftarrow x + 1$
(3) Stop
```

11) (1) $c \leftarrow a + b$, display c

(2) stop

12) (1) Enter x, y

(2) $z \leftarrow x + y$, display c

13 - 18[*] Make up a verbal algorithm that will:

13) Find the area of a rectangle given sides ℓ and w.

14) Find the perimeter and area of an equilateral triangle given side s.

15) Find the slope of the line between (a, b) and (c, d).

16) Find the sum of two rational numbers a/b and c/d.

17) Find the distance between two points (a, b) and (c, d).

18) Find the arithmetic mean $(\frac{x+y}{2})$ and geometric mean $(\sqrt{xy})$ of two positive real numbers x and y.

19) Using your algorithm from 17, write a program that will display the distance between any 2 points in the coordinate plane.

20) Find the distance between (-4, 5) and (17, -13) using your algorithm from (18), write a program that will display the arithmetic and geometric mean for any two positive real numbers. By comparing determine a relation between the geometric and arithmetic mean of two positive real numbers.

---

[*] Save these algorithms for further work in section 1.13.

1.10 Programming: Loops and Traces

Many algorithms contain loops; that is repeated series of steps. The long division algorithm you learned in fourth or fifth grade is an algorithm with a loop. You will recall that this (non-computer) algorithm goes something like this:

1. Divide $$\begin{array}{r} 2 \\ 23\overline{)4876} \end{array}$$

2. Multiply $$\begin{array}{r} 2 \\ 23\overline{)4876} \\ \underline{46} \end{array}$$

3. Subtract $$\begin{array}{r} 2 \\ 23\overline{)4876} \\ \underline{46} \\ 2 \end{array}$$

4. Bring down (the next digit) $$\begin{array}{r} 2 \\ 23\overline{)4876} \\ \underline{46} \\ 27 \end{array}$$

5. (same as 1) Divide $$\begin{array}{r} 21 \\ 23\overline{)4876} \\ \underline{46} \\ 27 \end{array}$$

6. (same as 2) Multiply $$\begin{array}{r} 21 \\ 23\overline{)4876} \\ \underline{46} \\ 27 \\ \underline{23} \end{array}$$

7. (same as 3) Subtract

and so on. Rather than repeat the same steps over and over this algorithm is expressed easily by a loop.

```
→ 1. Divide
│ 2. Multiply
│ 3. Subtract
│ 4. Bring down
└ 5. Go to 1.
```

In a similar way many computer-calculator algorithms and programs have loops to repeat steps or keystrokes. In the process usually the value(s) of one or more variables are changed and the steps are repeated.

EXAMPLE 1.10-1 Develop a verbal algorithm to generate successive powers of $a + bi$, for given a and b. Thus we want to calculate: $(a + bi)^1$, $(a + bi)^2$, $(a + bi)^3$, $(a + bi)^4$, etc., each answer in the form $e + fi$.

SOLUTION: We can develop this algorithm by modifying the algorithm of example 1.9-3. We "initialize" by setting $c + di$ (of example 1.9-3) $= 1 + 0i$

```
1. Remember $a$ and $b$.
2. $c \leftarrow 1$ and $d \leftarrow 0$
3. $e \leftarrow ac - bd$; display $e$    } Recall these steps
4. $f \leftarrow ad + bc$; display $f$    } from exercise 1.9-3
5. If the power is high enough, stop
6. $c \leftarrow e$ and $d \leftarrow f$
7. Go back to step 3.
```

To see how this algorithm operates, we develop what is called a trace, a record of the calculation through successive steps.

EXAMPLE 1.10-2 Develop a trace for example 1.10-1, when $a = -2$ and $b = 3$. This will generate successive powers of $-2 + 3i$.

First, set up a table with all letters represented in the algorithm:

a	b	c	d	e	f

Now we enter values of the program and follow the algorithm steps:

1. Remember a and b.

a	b	c	d	e	f
-2	3				

2. c ← 1 and d ← 0

a	b	c	d	e	f
-2	3	1	0		

3. e ← ac - bd; display e

a	b	c	d	e	f
-2	3	1	0	-2	

Note: e ← (-2)(3) - (3)(0)

4. f ← ad + bc; display f

a	b	c	d	e	f
-2	3	1	0	-2	3

Note: f ← (-2)(0) + (3)(1)

5. If power is high enough, stop

6. c ← e and d ← f

a	b	c	d	e	f
-2	3	~~1~~	~~0~~	-2	3
		-2	3		

7. Go back to step 3.

3. $e \leftarrow ac - bd$; display e

a	b	c	d	e	f
-2	3	~~1~~	~~0~~	~~-2~~	3
		-2	3	-5	

Note: $e \leftarrow (-2)(-2)-(3)(3) = -5$

4. $f \leftarrow ad + bc$; display f

a	b	c	d	e	f
-2	3	~~1~~	~~0~~	~~-2~~	~~3~~
		-2	3	-5	-12

Note: $f \leftarrow (-2)(3) + (3)(-2) = -12$

We have now calculated $(-2 + 3i)^2 = -5 -12i$

5. If the power is high enough, stop

6. $c \leftarrow e$ and $d \leftarrow f$

a	b	c	d	e	f
-2	3	~~1~~	~~0~~	~~-2~~	~~3~~
		~~-2~~	~~3~~	-5	-12
		-5	-12		

Of course in developing a trace, the table would only be drawn once.

Through $(-2 + 3i)^4$, the complete trace would appear as:

a	b	c	d	e	f	
-2	3	1	0	-2	3	$\Longrightarrow$ $(-2 + 3i)^1 = -2 + 3i$
		-2	3	-5	12	$\Longrightarrow$ $(-2 + 3i)^2 = -5 + 12i$
		-5	12	46	9	$\Longrightarrow$ $(-2 + 3i)^3 = 46 + 9i$
		46	9	-119	120	$\Longrightarrow$ $(-2 + 3i)^4 = -119 + 120i$

Quite often a vague step like step 5 is made subject to automatic control.

EXAMPLE 1.10-3 Develop an algorithm that calculates $(a + bi)^n$ as $e + f\,i$ for given a, b, and n.

```
1. Remember a, b, and n.
2. $c \leftarrow 1$, $d \leftarrow 0$, $k \leftarrow n-1$
3. $e \leftarrow ac - bd$
4. $f \leftarrow ad + bc$
5. If $k = 0$ display e and f and stop
6. $c \leftarrow e$, $d \leftarrow f$, and $k \leftarrow k-1$
7. Go back to step 3.
```

You will be asked to develop a trace for this example in the exercises.

Exercise Set 1.10

1) Develop a complete trace for $(1 - i)^3$ using the verbal algorithm of Example 1.10-3. Note that initial values are $a = 1$, $b = -1$, $n = 3$. Thus, after step 2 your trace will be:

a	b	n	k	c	d	e	f
1	-1	3	2	1	0		

2) From your trace in Exercise 1, give the value of $(1 - i)^3$.

3) Check your answer in Exercise 2 by multiplication.

4) Develop a complete trace for $(2i)^4$. Hint: $a = 0$, $b = 2$, $n = 4$.

5) Develop a complete trace for $(-1)^3$. Hint $b = 0$.

6-10. In these exercises you are asked to translate Example 1.10-3 verbal algorithm steps into calculator or computer steps for the device you have available. Exercise 6 is worked out as an example.

6) Remember a, b, and n (Algorithm step 1)

HP 33E: STO 0, R/S, STO 1, R/S, STO 2 (R_0=a, R_1= b, R_2= n)

TI 58: STO 00, R/S, STO 01, R/S, STO 02 (R_0 = a, R_1= b, R_2= n)

TRS-80: INPUT A, B, N

7) $c \leftarrow 1$, $d \leftarrow 0$, $k \leftarrow n-1$ (Algorithm step 2)

For calculators use $R_3 = c$, $R_4 = d$, $R_5 = k$. To calculate and store k use the sequence:

HP 33E: RCL 2, (why?), 1, -, STO 5

TI 58: RCL 02,(why?), -, 1, =, STO 05

TRS-80: K = N -1

8) $e \leftarrow ac - bd$ (Algorithm step 3)

For calculators use $R_6 = e$

9) $f \leftarrow ad + bc$ (Algorithm step 4)

For calculators use $R_7 = f$

10) $c \leftarrow e$, $d \leftarrow f$, and $k \leftarrow k-1$. (Algorithm step 6)

11) Go back to step 3. (Algorithm step 7)

Beware: Algorithm step 3 is probably not program step 3. (For calculators use $\boxed{\text{GTO}}$ the appropriate step number. For microprocessors use GO TO the appropriate line number.)

1.11 Decisions, decisions, decisions

In the exercises of section 1.10, you translated several verbal algorithm steps into program steps or key-strokes. In doing so we carefully avoided the program step for looping and the program step for making a decision. You will be asked in section 1.12 to refer to the operator's manual for your particular device to see how you would program loops. In this section we address the problem of making decisions.

Computers and now calculators are often referred to as "thinking machines." You may already have been impressed at the ability of your device to "think" its way through complex calculations. In fact you must admit already that your calculator is "ahead" of you: that is, it can do things you cannot do on your own. For example, your calculator can calculate in an instant the sine of 37° to many digits of accuracy. At this point in your education you probably cannot calculate that value at all!

But the most impressive thing your calculator can do is make decisions. In doing this it comes closest to mirroring (if not truly duplicating) human thinking. Of course the decisions your calculator can make are "simple" ones. Still many psychologists claim that all decisions, even the most complex ones, may be reduced to similar simple decisions.

Each computing device has its own special way of processing decisions. You will be asked in the exercises to review your operator's guide to determine how your own works. Here we describe two methods, the first common to many calculators, the second to many microprocessors and computers.

DECISION METHOD I SKIP ON FALSE

(This is a common calculator decision method.)

The calculator has one or more decision keys. The commonest one is $\boxed{x = 0\ ?}$, often labeled without the question mark. Other decision keys are:

	$x = y$?	$x = t$?
$x \neq 0$?	$x \neq y$?	$x \neq t$?
$x > 0$?	$x > y$?	$x > t$?
$x \geq 0$?	$x \geq y$?	$x \geq t$?

The latter keys save calculator arithmetic but two such keys would be sufficient.

In a program the way these keys operate is amazingly simple:

(1) If the answer to the question is TRUE, the calculator continues on to the next program step

(2) If the answer to the question is FALSE, the calculator skips the next program stop.

EXAMPLE 1.11-1 In the program sequence

```
 .
 .
 .
09    x > 0 ?
10    R/S
11    1/x
12    R/S
 .
 .
```

What would happen if the x-register (the display) after step 8 was (a) 5, (b) -2, or (c) 0.

SOLUTION:

(a) x = 5, x >0 is TRUE so the calculator would stop and display 5.

(b) x = -2, x >0 is FALSE so the calculator would skip to step 11, calculate the reciprocal, -0.5 and (in step 12) stop to display that result.

(c) x = 0, x > 0 is FALSE so the calculator would skip to step 11, try to calculate the reciprocal and fail, halting the calculator to display an error message.

DECISION METHOD II CONTINUE ON FALSE

(This is a common computer decision method.)

The more sophisticated computer languages usually have IF ... THEN statements (as well as more complex decision statements.) Here again the processing is very simple:

(1) If the answer is TRUE, carry out the instruction following THEN

(2) If the answer is FALSE, go on to the next program step.

EXAMPLE 1.11-2 In the program sequence

```
09    IF x > 0,  THEN (GO TO) 12
10    x = 1/x*
11    PRINT x
12    STOP
```

What would happen if the X-register at step 8 were (a) 5, (b) -2, or (c) 0.

---

[*] In many computers x = 1/x is machine language for x ← 1/x, that is replace x by 1/x. This is a different (and algebraically incorrect) use of the = sign.

SOLUTION:

(a) X= 5, the computer would stop.

(b) X = -2, the computer would go on to step 10, calculate 1/x = -0.5, print this value in step 11 and stop at step 12.

(c) X = 0, the calculator would go on to step 10 and at that point stop to give an error message.

Exercise Set 1.11

1 - 6 Determine what the calculator would stop and display in each case proceeding through these program steps when the value at step 5 is as given:

```
06      x ≥ 0
07      GTO 10*
08      x²
09      R/S
10      1/x
11      R/S
```

1) $x = -2$ 2) $x = 5$ 3) $x = 2$

4) $x = -1$ 5) $x = 100$ 6) $x = 0$

7 - 12 Determine what the calculator would stop and display when the value at step 5 is as given:

```
06      x ≥ 0
07      GTO 09*
08      x²
09      1/x
10      R/S
```

Hint: The results are not all the same as exercises 1 - 6.

---

[*] By this instruction we mean here for the calculator to jump to the step number given (not to the label given).

7) $x = -2$ 8) $x = 5$ 9) $x = 2$

10) $x = -1$ 11) $x = 100$ 12) $x = 0$

13 - 18 Determine what the computer would print when the value at step 5 is as given:

```
06   IF  X ≥ 0  THEN (GO TO)  10
07   X  =  X + 1
08   PRINT  X
09   STOP
10   PRINT  -X
11   STOP
```

13) $X = -2$ 14) $X = 5$ 15) $X = 2$

16) $X = -1$ 17) $X = 100$ 18) $X = 0$

19 - 24 Determine what the computer would print when the value at step 5 is as given:

```
06   IF  X ≥ 0  THEN (GO TO)  08
07   X = X + 1
08   PRINT  -X
09   STOP
```

19) $X = -2$ 20) $X = 5$ 21) $X = 2$

22) $X = -1$ 23) $X = 100$ 24) $X = 0$

25 - 26 The following is a verbal algorithm for a pocket calculator.*

```
10   Empty all your pockets

20   A ← 0

30   Take a piece of paper and number it  A

40   If you have no empty pockets then (GO TO)  80

50   Place paper numbered  A  in an empty pocket
```

---

* This algorithm was submitted by David Lloyd in the September, 1978 issue of a British journal called _Mathematics Teaching._

```
60    A ← A + 1
70    GO TO 30
80    Read this piece of paper
90    STOP
```

25) What is the decision that is made?

26) What does the algorithm do?

1.12 Your Own Calculating Device

For any programming you wish to do it is extremely important that you know the quirks and idiosyncrasies of the calculator, microprocessor, or computer you are using. To familiarize yourself with your particular device, your best recourse is the operator's manual or programming guide for that device. Even if you feel very familiar with the calculating device you are using, you will wish to consult such a guide in answering some of the following questions.

Exercise Set 1.12

Answer the following questions for the specific calculating device (or devices) you will be using as you study this text.

1) Is your device a programmable calculator, a microprocessor, or a computer terminal?

2) Does your device calculate in AH or RPN or a third operating order?

3) What switch turns your device on and off?

4) Are programs "lost" when you turn your device off?

5) If your answer to exercise 4 was "yes", how can you retain a program so that you don't have to work out the details the next time you need it?

6) Does your device have printing capability? If so, how do you signal it to print?

7) How many storage locations does your device have? How are they named?

8) What happens if you have a number (say 5) stored in a register and then store a new number (say 7) in that same register?

9) Does your device have register arithmetic or storage arithmetic? That is, can you modify what is in a storage register without recalling that value, operating on it, and restoring it. If your device has register arithmetic, describe how you would use it to

(a) add 3 to R_5

(b) subtract 2 from R_5

(c) multiply the numbers in R_1 and R_2 and store the product in R_2.

(d) divide the number in R_4 by the number in R_3 and store the quotient in both R_3 and R_4.

(e) multiply R_5 by zero. How else could you accomplish this?

10) How do you instruct your device to accept a program?

11) How do you change from program to calculating mode if it is necessary to do this?

12) Does your device have a way of labeling programs so that you can enter more than one at a time?

13) How do you set your device at the beginning of a program ready to process that program?

14) How does your device loop? What is its basic instruction to accomplish this?

15) Do you loop to a program step or a label?

16) Which keys on your device do not enter program steps (when in programming mode)?

17) How can you review the steps in a program that are keyed into your device?

18) Does your device allow replacement of a program step? If so, how do you accomplish this?

19) Does your device allow you to delete a program step? If so, how do you accomplish this?

20) Does your device allow you to insert a new program step in the middle of a program? If so, how do you accomplish this?

1. 13 Using Your Own Calculating Device

The very best way for you to become familiar with your calculator, microprocessor or computer is to use it. The most interesting way to use your equipment is to solve problems. Throughout this chapter we have presented verbal algorithms and programs that have solved specific problems, like multiplying two complex numbers.

Exercise set 1. 13 contains more problems that you can use to help you understand the operation of your calculating device. Each of these problems can be solved in many ways with solutions that range from very simple to extremely sophisticated. We suggest that you start with a verbal algorithm and a simple solution. As your familiarity with your equipment increases you can broaden the focus of your attention beyond the solution of the problem to solving the problem in the fewest steps or displaying additional information. Make sure that you test each program with several trials whose answers you know (or are willing to compute by hand.)

Exercise Set 1. 13

1 - 4 Refer back to your solutions for exercises (13 - 16) from section 1. 9.

Write a program to find:

1) The area and perimeter of a rectangle given sides ℓ and w.

2) The area and perimeter of an equilateral triangle given side s.

3) The slope of the line between (a, b) and (c, d).

4) The sum of two rational numbers a/b and c/d.

5 - 10 These problems refer to mathematical ideas that you have seen, but have perhaps forgotten. You may want to look back at your old notes (if they still exist) or another math book.

5) Given the hypotenuse and one acute angle of a right triangle find
   (a) the measure of the other acute angle.
   (b) the length of each leg.
   (c) the perimeter of the triangle.
   (d) the area of the triangle.

6) Given the equation of a parabola, $f(x) = ax^2 + bx + c$ find
   (a) the vertex (turning point)
   (b) the equation of the axis of symmetry
   (c) the sum of the roots
   (d) the product of the roots

   For (c) and (d) do not compute the roots.

7) Find the area of a triangle given two sides and the included angle.

8) Find the area of a triangle given three sides. You might want to use (or find) Hero's formula.

9) Given two points in the plane, determine the equation of the line through the two points.

10) Find all the primes less than 100.

11) Find the sum of the squares of N consecutive positive integers.

12) Given any integer less than 100, find
   (a) its smallest prime factor
   (b) all its prime factors

1.14 Chapter 1 Test

(1 - 4) Give a keystroke sequence that you could use on your calculator to evaluate each of the following expressions.

1) $5 + 6/7$ 2) $(3 + 4)(5 + 7)$

3) $\sqrt[3]{6 \sin 37^{\circ}}$ 4) $\frac{2 + 3}{5 + 7}$

(5 - 6) Evaluate each of the following.

5) $2 + 5 \div 5 + 7$ 6) $2 \div 5 + 5 \times 7$

(7 - 9) Determine what the calculator would stop and display in each case proceeding through these program steps when the value at step 5 is as given.

```
06    x ≥ 0
07    GTO 11
08    CHS (or +/-)
09    √x
10    R/S
11    √x
12    R/S
```

7) x = -4 8) x = 2 9) x = 0

(10 - 12) Determine what the computer would print when the value at step 5 is as given.

```
06    If x ≤ 0 then(go to) 10
07    x = x ÷ 2
08    PRINT x
09    STOP
10    x = x²
11    PRINT x
12    STOP
```

10) x = -.6 11) x = 0 12) x = 3

(13 - 14) Evaluate each of the following expressions. Round your answer to the nearest hundredth.

13) $\dfrac{\sqrt{8.7} + \sin 48.6^\circ}{(1.5)^2 - \dfrac{1}{3.2}}$

14) The area of a triangle with sides a, b and c is given by Hero's Formula:

$A = \sqrt{s(s-a)(s-b)(s-c)}$

where $s = 1/2(a+b+c)$.

Find the area of a triangle whose sides are 6.2, 3.6, and 5.1

(15) The time needed to complete one period of a pendulum is given by the formula $t = \pi\sqrt{\dfrac{\ell}{9.8}}$ where ℓ represents the length of the pendulum. Find the value of ℓ, correct to the nearest hundredth, that makes t closest to 3.7 seconds.

(16) Using the formula $F = \frac{9}{5}C + 32$:

a) Write a verbal algorithm to convert a Centigrade temperature to a Fahrenheit temperature.

b) Write a program for your calculating device that uses your algorithm from a).

c) Set your calculating device to exhibit answers rounded to the nearest integer. Using your program from b) find a temperature where the digits of the Fahrenheit temperature are reversed to represent the Centigrade temperature. Example:

$$45^\circ C \overset{?}{=} 54^\circ F \quad \text{or} \quad 68^\circ C \overset{?}{=} 86^\circ F$$

(17 - 18) Choose one of the following two questions.

17) We wish to calculate the real and imaginary coefficients of the reciprocal of a complex number.

a) Algebraically determine the real and imaginary coefficients of $\dfrac{1}{a + bi}$.

b) Write a verbal algorithm to calculate the real and imaginary coefficients of the reciprocal of a complex number.

c) Write your algorithm from b) as a program for your calculating device.

(d - g) Using c), rewrite each of these complex numbers in standard form (that is as $c + di$).

d) $\frac{1}{2 + 3i}$ e) $\frac{1}{-1 + i}$ f) $\frac{1}{i}$ g) $\frac{1}{5 - 7i}$

(18) As already mentioned in question (14) that area of a triangle with sides a, b and c is given by the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where } s = \frac{1}{2}(a + b + c)$$

a) Write a verbal algorithm to determine the area of a triangle given three sides, using Hero's Formula.

b) Write a program that will find the area of a triangle using your algorithm from a).

(c - d) Load your program into your calculating device and find the area of each of the following triangles to the nearest integer.

c) $a = 88$, $b = 72$, $c = 108$ d) $a = 2$, $b = 3$, $c = 5$

e) Explain your answer to d).

CHAPTER 2. SEQUENCES, SERIES, AND LIMITS

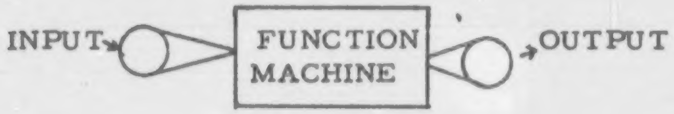
In this chapter you will have the opportunity to use your new-found programming power together with your more formal mathematical skills to gain insights into some important new ideas.

Ever since you learned to count you have dealt with the kinds of patterns that lead to sequences. We will now examine these and more complex sequences. You will find that they lead us to some important applications as diverse as compound interest and the length of the path of a bouncing ball.

2.1 Sequences

A sequence, S, is a function whose domain is the natural numbers.

Let us examine what this formal definition means. We know what a function is. It is a "mathematical machine" that accepts certain input numbers[*] and produces from them output numbers[*] as in the diagram.



We also know what the natural (or counting) numbers are. They are the numbers 1, 2, 3, 4, 5, and so on. Our definition of a sequence then is a function whose input numbers are restricted to 1, 2, 3, 4, 5 and so on, no negative or fractional or irrational numbers allowed.

---

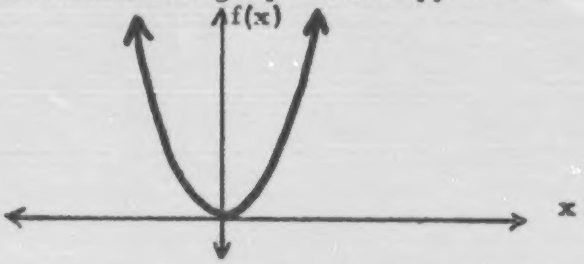
[*] or other mathematical elements

EXAMPLE 2.1-1 What is the difference between the squaring function and the sequence of squares?

Solution: The squaring function $f(x) = x^2$ or $f: x \rightarrow x^2$ has as its domain all real numbers. Thus a table of values could include the following:

x	x^2
$1/2$	$1/4$
-3	9
0	0
$\sqrt{2}$	2
3.7	13.69

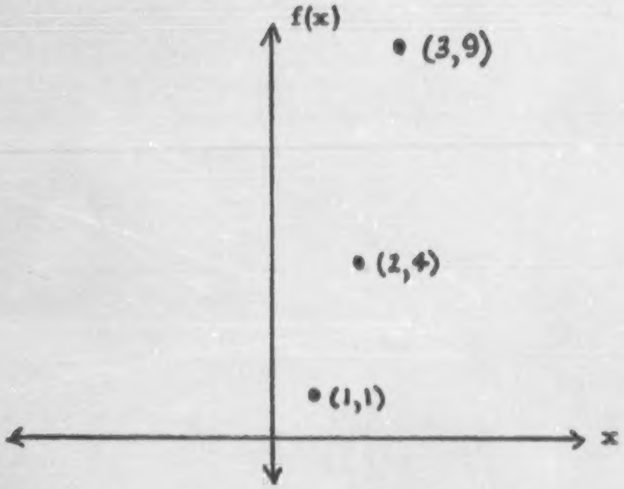
and the familiar graph would appear as:



The sequence table would be more restricted

x	x^2
1	1
2	4
3	9
4	16
.	.
.	.
.	.

and its graph would include only isolated points.



In the example our sequence of squares, S, could be listed as the set of ordered pairs:

$$S = \{(1, 1), (2, 4), (3, 9), (4, 16), \ldots\}$$

the 3 dots meaning "and so on" following the same pattern.

Sometimes the general term is also named explicitly:

$$S = \{(1, 1), (2, 4), (3, 9), \ldots, (n, n^2), \ldots\}$$

You will be pleased to know that this complex notation is usually abbreviated by listing only the outputs in order:

$$S = \{1, 4, 9, \ldots, n^2, \ldots\}$$

or even

$$S = (n^2)\ ^*$$

In these notations n is assumed to represent a natural number.

When we wish to speak of a particular term of a sequence, again several notations are possible. The third term of the sequence S may

be referred to as $S(3)$

or S_3 .

---

* The use of parentheses instead of brackets here is common sequence notation.

In the sequence of squares, $S_3 = 9$. We could specify this sequence by designating the n^{th} term:

$$S_n = n^2.$$

A sequence is called finite if it has a finite number of terms: more simply, it stops. A sequence is called infinite if it is not finite, that is it "goes on forever."

EXAMPLE 2.1-2 For the sequence defined by

$$S_n = \sqrt{2n+1}$$

give the first five terms.

Solution: Substitute 1, 2, 3, 4, and 5 in the defining function to obtain:

$$\sqrt{3}, \quad \sqrt{5}, \quad \sqrt{7}, \quad 3, \quad \sqrt{11}$$

Notice in example 2.1-2 that while inputs must be natural numbers, outputs are not so restricted.

Exercise Set 2.1

1 - 9 For each sequence as defined, give the first six terms:

1) $S_n = 2n - 1$

2) $(n^2 - n)$

3) $(1/n)$

4) (2^n)

5) $S_n = n^n$

6) $(\sqrt[n]{n})$

7) $S_n = (-1)^n$

8) $\left(\frac{n}{n+1}\right)$

9) $S_n = \frac{3}{10^n}$

10 - 18 For each sequence, give the required term:

10) $S_n = 2n-1,\ S_{40}$

11) $(n^2-n),\ S_{100}$

12) $(\frac{1}{n}),\ S_{50}$

13) $(2^n),\ S(20)$

14) $S_n = n^n,\ S(40)$

15) $(\sqrt[n]{n}),\ S_{25}$

16) $S_n = (-1)^n,\ S(100)$

17) $\left(\frac{n}{n+1}\right),\ S_{99}$

18) $S_n = \frac{3}{10^n},\ S(40)$

19 - 22 A verbal algorithm for generating successive values of a sequence is:

```
1.  Set $n = 1$.
2.  Evaluate $S_n$.
3.  Display $n$, $S_n$.
4.  If you have enough values,  stop.
5.  $n \leftarrow n + 1$.
6.  Go to step 2.
```

Program your calculating device for the sequence

$$S_n = \frac{5n + 1}{n}$$

19) Run your program to give the first five terms of S_n.

20) Continue to run your program until you can predict a number S_{1000} will be close to.

21) Does $\frac{5n + 1}{n} = 5 + \frac{1}{n}$?

22) How does your answer in exercise 21 help to justify your answer in exercise 20?

23 - 26 Program your calculating device to generate successive terms of the sequence defined by

$$S_n = \frac{n^3}{2^n}$$

23) Give the first six terms.

24) Which term is largest? Give n and S_n.

25) Run your program until you can predict a number S_{1000} will be close to.

26) Use the following representation for the first seven terms of this sequence to explain your answer in exercise 25:

$$\frac{1\cdot 1\cdot 1}{2}, \frac{2\cdot 2\cdot 2}{2\cdot 2}, \frac{3\cdot 3\cdot 3}{2\cdot 2\cdot 2}, \frac{4\cdot 4\cdot 4}{2\cdot 2\cdot 2\cdot 2}, \frac{5\cdot 5\cdot 5}{2\cdot 2\cdot 2\cdot 2\cdot 2}, \frac{6\cdot 6\cdot 6}{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2}, \frac{7\cdot 7\cdot 7}{2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 2}$$

2.2 Arithmetic Sequences

In section 2.1 you met a variety of sequences. In order to sort these sequences into categories mathematicians assign several names to types. Some examples are:

Defining function	First terms	Type
$S_n = 7$	7, 7, 7, 7, ...	Constant sequence
$S_n = (-1)^n$	-1, 1, -1, 1, -1, ...	Alternating sequences (alternating signs)
$S_n = (-2)^{n-1}$	1, -2, 4, -8, 16, ...	

A more important type of sequence because it has a number of interesting applications is an arithmetic sequence. An arithmetic sequence is also known as a linear sequence or an arithmetic progression.

> An arithmetic sequence is a sequence in which the difference between successive terms, d, is constant.[*]

Here are several arithmetic sequences:

Defining function	First terms	Common difference
$S_n = n$	1, 2, 3, 4, 5, ...	1
$S_n = 10-2n$	8, 6, 4, 2, 0, -2, ...	-2
$(2n - 1)$	1, 3, 5, 7, ...	2

It is not enough for some differences to be the same; all must be alike.

---

[*] Although we have not stated it explicitly in our definition, order is important. By the difference between successive terms we mean $S_{n+1} - S_n$.

EXAMPLE 2.2-1 Is $(n^3 - 6n^2 + 12n)$ an arithmetic sequence?

Solution: $S_1 = 7$

$S_2 = 8$

$S_3 = 9$

These three values suggest an arithmetic sequence with $d = 1$. But $S_4 = 16$

$S_5 = 35$

$S_6 = 72$

Clearly the difference is no longer constant and this is not an arithmetic sequence.

Since we have defined an arithmetic sequence as having a common difference, d, we can represent successive terms as follows:

$$S_2 = S_1 + d$$

$$S_3 = S_2 + d = (S_1 + d) + d = S_1 + 2d$$

$$S_4 = S_3 + d = (S_1 + 2d) + d = S_1 + 3d$$

In the same way:

$$S_5 = S_1 + 4d$$

$$S_6 = S_1 + 5d$$

$$\vdots$$

In general we can write:

$$\boxed{S_n = S_1 + (n-1)d}$$

This formula would be of little importance if we always had the defining function for S_n. We often do not have that function stated. Instead we may be given the first several terms as in the following examples:

EXAMPLE 2.2-2 Find the 20th term (S_{20}) for the arithmetic sequence whose first few terms are:

$$\frac{1}{2},\ 2,\ 3\frac{1}{2},\ 5,\ 6\frac{1}{2},\ \dots$$

Solution: We identify $S_1 = \frac{1}{2}$, $n = 20$, and $d = 1\frac{1}{2}$.

Substituting in our formula we have

$$S_{20} = \frac{1}{2} + (20 - 1)\,1\frac{1}{2} = \frac{1}{2} + (19)\,(1\frac{1}{2}) = \frac{1}{2} + 28\frac{1}{2} = 29.$$

EXAMPLE 2.2-3 If I purchase a bicycle for a $50 down payment and $10 per month for a year, how much will I have paid at the end of the year?

Solution: We translate the total paid so far into a sequence

1st month	60
2nd month	70
3rd month	80

We have an arithmetic sequence with $S_1 = 60$, $d = 10$, $n = 12$

$$S_{12} = 60 + 11 \cdot 10 = 170$$

The total cost of the bike is $170.

Exercise Set 2.2

1 - 6 Which of the following sequences are arithmetic? For arithmetic sequences find d.

1) 27, 25, 23, 21, . . .
2) $S_n = n^2$
3) $(3n - 5)$
4) 4, 4, 4, 4, 4, . . .
5) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
6) 1.6, 0.7, -0.2, -1.1, . . .

7 - 10 Give the first five terms:

7) $S_1 = 3, \quad d = 4$
8) $S_1 = 5, \quad d = -7$
9) $S_1 = p - 2q, \ d = q$
10) $S_3 = 6, \ d = 2$ Be careful!

11 - 14 Use the formula $S_n = S_1 + (n-1)d$ to find the indicated term for each arithmetic sequence:

11) 1, 3, 5, . . .; S_{21}
12) 10, 7, 4, 1, . . .; S_{15}
13) 1, 2, 3, 4, . . .; S_{20}
14) 50, 52.50, 55, . . .; S_9

15) In terms of d the difference between S_6 and S_9 is 3d. Why?

16 - 17 Express in terms of d the difference between

16) S_1 and S_5
17) S_{35} and S_{50}

18) The arithmetic mean between two numbers is the term that would come between the two numbers if they were in arithmetic progression. Thus finding the arithmetic mean between 12 and 37 means letting $S_1 = 12$, $S_3 = 37$ and finding S_2. Use what you found in exercises 15 - 17 to find d and then S_2.

19) Find the arithmetic mean between 10 and 25.

20) Find the arithmetic mean between a and b.

21) Use your answer in 20 to justify the use of the synonym average for the arithmetic mean.

22 - 23 To insert n terms between two given terms of an arithmetic sequence is often (misleadingly) called finding n arithmetic means between the two numbers. Generalize your method in exercise 18 to find the following:

22) Three arithmetic sequence terms between 5 and 13.

23) Five arithmetic means between 37 and 19.

24) Joe saves \$2000 for college spending money. He plans to spend \$50 per week of this. Use the arithmetic sequence formula to determine how much he has left after 26 weeks. (Be careful. Note that $S_1 = 1950$!)

25) Find a formula for the amount Joe has left after w weeks (in exercise 24).

26) In an arithmetic sequence $S_3 = 40$ and $S_{10} = 12$; find S_{17}. Hint: first find d.

27 - 30 An arithmetic sequence is also called a linear sequence because its defining function is linear; that is, of the form

$$mx + b.$$

Thus the defining function a linear or arithmetic sequence takes the form

$$(mn + b) \text{ or } S_n = mn + b$$

27) Find S_1 as a function of m and b.

28) Find $d = S_2 - S_1$ as a function of m and b.

29) Use what you found in exercises 27 and 28 to determine the defining function for the sequence: 5, 7, 9, 11, ...

30) Find the defining function for the sequence $\frac{1}{4}, 1, 1\frac{3}{4}, 2\frac{1}{2}, \ldots$

2.3 Geometric Sequences

Geometric sequences are also called geometric progressions or exponential sequences.

> A geometric sequence is a sequence in which there is a constant ratio, $r \neq 0$, between successive terms.*

Here are some geometric sequences:

Defining function	First terms	Common ratio
$S_n = 2^{n-1}$	1, 2, 4, 8, 16, ...	2
(0.5^n)	0.5, 0.25, 0.125, 0.0625, ...	0.5
$\left(\frac{5^{n-1}}{100}\right)$	$\frac{1}{100}, \frac{1}{20}, \frac{1}{4}, \frac{5}{4}, \frac{25}{4}, \frac{125}{4}, \ldots$	5
$S_n = (-1)^n$	-1, 1, -1, 1, -1, 1, ...	-1
(3)	3, 3, 3, 3, 3, 3, ...	1

Notice that the last (constant) sequence also has a common difference, $d = 0$, so it is both a geometric and an arithmetic sequence.

From the defining relationship for a geometric sequence

$$\frac{S_n + 1}{S_n} = r$$

we may obtain

$$S_{n+1} = r\,S_n.$$

---

* As in the case of the arithmetic sequence, order is important here; thus $r = (S_{n+1})/S_n$.

Applying this to the first few terms, we have

$$S_2 = rS_1$$

$$S_3 = rS_2 = r(rS_1) = r^2S_1$$

$$S_4 = rS_3 = r(r^2S_1) = r^3S_1$$

Similarly:

$$S_5 = r^4S_1$$

$$S_6 = r^5S_1$$

This pattern leads to the general formula for S_n.

$$\boxed{S_n = r^{n-1}\, S_1}$$

EXAMPLE 2.3-1 Find the ninth term of the sequence whose first terms are 3, 12, 48, 192, ...

Solution: We identify $S_1 = 3$, $r = \frac{12}{3} = 4$, and $n = 7$.

$$S_7 = 4^{7-1} \cdot 3 = 4^6 \cdot 3 = 12{,}288.$$

To calculate products like these, a calculator is very useful. In the next example it is even more important.

EXAMPLE 2.3-2 In a geometric sequence $S_3 = 4.056$ and $S_7 = 11.5843416$, find S_1 and S_5.

Solution: $S_3 = r^2S_1$ so (1) $r^2S_1 = 4.056$

$S_7 = r^6S_1$ so (2) $r^6S_1 = 11.5843416$

Dividing (2) by (1) we have:

$$r^4 = 2.8561 \text{ and } r = \sqrt[4]{2.8561}$$

And by calculator (using the $\boxed{y^x}$ key or $\boxed{\sqrt{\ }}$ $\boxed{\sqrt{\ }}$):

$$r = 1.3$$

To find S_1, substitute this value of r into (1):

$$(1.3)^2 S_1 = 4.056$$

$$S_1 = 2.4$$

To find S_5: $S_5 = (1.3)^4 \cdot 2.4 = 6.85464$

Exercise Set 2.3

1 - 6 Identify which of the following are geometric sequences. For those that are geometric, find r.

1) 1, 5, 25, 125, ...
2) 8, 2, $\frac{1}{2}$, $\frac{1}{8}$, ...
3) 2, 6, 18, 72, ...
4) 144, 120, 100, $83\frac{1}{3}$
5) 0, 0, 0, 0, ...
6) 1, -3, 9, -27, 81, ...

7 - 10 Give the first five terms of each geometric sequence

7) $S_1 = 3$, $r = 2$
8) $S_1 = 1$, $r = 0.6$
9) $S_1 = a$, $S_2 = ab$
10) $S_3 = a$, $r = b$

11 - 14 Find the required term of each geometric sequence:

11) 1024, 512, 256, ...; S_{12}
12) 3, 2.4, 1.92, ...; S_6
13) $S_1 = 72$, $S_3 = 8$; S_8
14) $S_3 = 18$, $S_7 = 91.125$; S_2

15 - 18 The geometric mean between two numbers is the number that would fall between them in a geometric sequence. Thus to find the geometric mean between 2 and 32, let $S_1 = 2$ and $S_3 = 32$, and find S_2. Use this procedure to find the geometric mean between each of the following:

15) 2, 32 16) 32, 2

17) a, a^3 18) a, b

19) Find the mean proportional between a and b. That is, solve the proportion

$$\frac{a}{g} = \frac{g}{b}$$

for g.

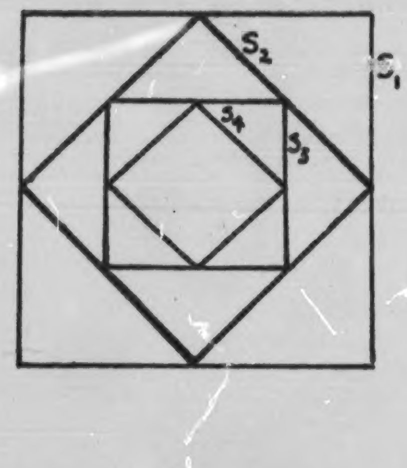
20) What do your results in exercises 19 and 20 tell you about the mean proportional and the geometric mean?

21 - 22. Inserting more than one geometric mean between two numbers is like the process for arithmetic means. To insert three means between 7 and 567, for example, set $S_1 = 7$, $S_5 = 567$, and find S_2, S_3, and S_4.

21) Insert three geometric means between 7 and 567.

22) Insert two geometric means between 567 and 168.

23) A golfball dropped on a cement floor bounces 80% as high. If the ball is dropped from a height of 2 meters, give the heights to which it boundes for the first 5 bounces. Is this a geometric sequence? If so what is r?

24) 

A "nest" of squares may be constructed by joining midpoints of sides. Show that the areas of the squares form a geometric sequence; that is, that $S_{n+1} = r\,S_n$. (Find r.) If the area S_1 is 96, find S_7.

2.4 Finite Series

We found in sections 2.1 to 2.3 how to calculate specific terms of many sequences. We now turn our attention to finding the sum of the terms of sequences.

Clearly there is a direct way of doing this.

EXAMPLE 2.4-1 Find the sum of the first seven terms of the arithmetic sequence $(2 + 3n)$

Solution: Generate the first seven terms: 5, 8, 11, 14, 17, 20, 23. Add them to get 98.

EXAMPLE 2.4-2 Find the sum of the first eight terms of the geometric sequence 3, 6, 12, ...

Solution: $r = 2$, so the eight terms are: 3, 6, 12, 24, 48, 96, 192, 384. Add to get 765.

We have used the notation S_n or $S(n)$ for the n^{th} term of a sequence. We now introduce the symbol $\$_n$ (using the regular dollar sign) to represent the sum of terms up to S_n.

Thus:

$$\$_1 = S_1$$

$$\$_2 = S_1 + S_2$$

$$\$_3 = S_1 + S_2 + S_3$$

$$\$_4 = S_1 + S_2 + S_3 + S_4$$

and in general

$$\$_n = S_1 + S_2 + S_3 + \ldots + S_n$$

Applying this notation to the examples, in example 2.4-1 we found $\$_7 = 98$ and in example 2.4-2 we found $\$_8 = 765$

It is important to recognize that $\$_1, \$_2, \$_3, \$_4, \ldots$ itself forms a sequence. This sequence of sums is called a series.

Whenever we are given the defining function for a sequence (the formula for S_n), we can use a simple algorithm to calculate $\$_n$. This verbal algorithm is a minor modification of the program of section 2.1 (see exercise 2.1 - 7.) used to calculate sequence terms. We will display both S_n and $\$_n$.

```
TO CALCULATE $S_n$, and $\$_n$ for $n = 1, 2, 3, \ldots$

    1. Set $n \leftarrow 1$, $\$ = 0$
    2. Calculate $S_n$
    3. Display $n$, $S_n$
    4. $\$ \leftarrow \$ + S_n$
    5. Display $\$$
    6. If $n$ is large enough, stop.
    7. $n \leftarrow n + 1$
    8. Go to step 2.
```

This verbal algorithm has several problems associated with it. First, it does not work for sequences for which the general term is not available. You will be asked to modify the program to take care of this for arithmetic and geometric sequences. Another problem is that the process is inefficient. For example, if we want to calculate $\$100$, we must first calculate $\$_1$ to $\$_{99}$.

In the case of many series, particularly the arithmetic and geometric series, we can calculate $\$_n$ directly. We now turn to the development of those formulas. In each case we start from the basic relationship

$$(1) \qquad \$_n = S_1 + S_2 + S_3 + \dots + S_{n-2} + S_{n-1} + S_n$$

and employ a trick.

The Arithmetic Series

We recall the role of d: $S_{n+1} = S_n + d$ for all n in an arithmetic sequence. Thus we may substitute in (1) to obtain:

$$(2) \quad \$_n = S_1 + (S_1 + d) + (S_1 + 2d) + \dots + (S_n - 2d) + (S_n - d) + S_n$$

Now we use the tricks. We rewrite the right member of (2) from right to left

$$(3) \quad \$_n = S_n + (S_n - d) + (S_n - 2d) + \dots + (S_1 + 2d) + (S_1 + d) + S_1$$

and then add (2) and (3)

$$(4) \quad 2\$_n = (S_1 + S_n) + (S_1 + S_n) + (S_1 + S_n) + \dots + (S_1 + S_n) + (S_1 + S_n)$$
$$+ (S_1 + S_n)$$

On the right side of (4) we have n pairs (Do you see why?) so:

$$(5) \quad 2\$_n = n(S_1 + S_n)$$

and

$$(6) \quad \boxed{\$_n = \frac{n}{2}(S_1 + S_n).}$$

Now we have, in equation (6), a formula for the sum of a sequence, \$, given the first and last terms, S_1 and S_n and the number of terms, n.

If we don't know S_n, we can calculate it by the formula of section 2.2,

$$(7) \qquad \boxed{S_n = S_1 + (n-1)d,}$$

or we can substitute this for Sn in (6) to give:

$$(8) \quad \boxed{\$_n = \frac{n}{2}[2S_1 + (n-1)d]}$$

EXAMPLE 2.4-3 Find the sum of the first 20 terms of the sequence beginning: 5, 8, 11, ...

Solution: $S_1 = S$, $d = 3$, $n = 20$. Substituting in formula (8):

$$\$_{20} = \frac{20}{2}[2 \cdot 5 + (20-1)3] = 670$$

EXAMPLE 2.4-4 Find the sum of the first 100 natural numbers.

Solution: $S_1 = 1$, $S_{100} = 100$, $n = 100$, $d = 1$. Using formula (6):

$$\$_{100} = \frac{100}{2}(1 + 100) = 5050$$

The Geometric Series

Turning now to the geometric sequence we recall the role of r: $S_{n+1} = r\,S_n$ for all n. We substitute this in (1) to obtain:

$$(9) \quad \$_n = S_1 + rS_1 + r^2S_1 + \ldots + r^{n-3}S_1 + r^{n-1}S_1$$

Now the trick. Multiply (9) by r:

$$(10) \quad r\,\$_n = rS_1 + r^2S_1 + r^3S_1 + \ldots + r^{n-2}S_1 + r^{n-1}S_1 + r^nS_1$$

Subtract (10) from (9):

$$(11) \quad \$_n - r\$_n = S_1 - r^nS_1.$$

Factor the left member of (11):

$$(12) \quad \$_n(1-r) = S_1 - r^nS_1$$

This leads to

$$(13) \quad \boxed{\$_n = \frac{S_1 - r^n S_1}{1 - r}} \quad \text{or} \quad (14) \quad \boxed{\$_n = \frac{S_1(1 - r^n)}{1 - r}}$$

This formula corresponds to the second arithmetic sequence formula, (8). We can rewrite (13) as

$$(15) \quad \$_n = \frac{S_1 - r(r^{n-1}S_1)}{1 - r}$$

and substitute Sn for $r^{n-1}S_1$ to give a formula corresponding to (7):

$$(16) \quad \boxed{\$_n = \frac{S_1 - rS_n}{1 - r}}$$

EXAMPLE 2.4-5 Find the sum of the first 10 terms of the geometric sequence whose defining function is $(3 \cdot 2^n)$.

Solution: The first several terms are 6, 12, 24, ... $S_1 = 6$, $r = 2$ and $n = 10$. Substituting in (14):

$$\$_{10} = \frac{6(1 - 2^{10})}{1 - 2} = \frac{6 \cdot (-1023)}{-1} = 6138.$$

We summarize here the formulas we have developed:

ARITHMETIC SEQUENCES	GEOMETRIC SEQUENCES
$S_n = S_1 + (n-1)d$	$S_n = S_1 r^{n-1}$
$\$_n = \frac{n}{2}(S_1 + S_n)$	$\$_n = \frac{S_1 - rS_n}{1 - r}$

Arithmetic Sequences (cont.)

$$\$_n = \frac{n}{2}\left[2S_1 + (n-1)d\right]$$

Geometric Sequences (cont.)

$$\$_n = \frac{S_1 - r^n S_1}{1 - r}$$

$$\$_n = \frac{S_1(1 - r^n)}{1 - r}$$

Exercise Set 2.4

For many exercises a calculator will be helpful.

1) Find the sum of the first 50 natural numbers.

2) Find the sum of the first 20 powers of 2: that is, $2^1, 2^2, 2^3, \ldots, 2^{20}$.

3 - 10 Identify each sequence as arithmetic, geometric, or other. Find the required $\$_n$.

3) $1, 3, 5, 7, \ldots; \$_{20}$

4) $1, 3, 5, 7, \ldots; \$_n$

5) $8, 4, 2, 1, \ldots; \$_{10}$

6) $S_n = 1000(0.5)^n$; $\$_5$

7) $(5 - 3n)$; $\$_{20}$

8) $(81 (\frac{2}{3})^n)$; $\$_7$

9) $(x^2 - x + 3)$; $\$_5$

10) $800, 750, 700, \ldots; \$_{21}$

11) In a famous problem you are asked which you would prefer to receive: a single payment of a million dollars or a 30 day month in which you are paid 1¢ the first day, 2¢ the second, 4¢ the third, and so on, doubling the amount you are paid on each succeeding day. Determine the better offer by careful calculation.

12) Program and solve exercise 3 by following the verbal algorithm of this section.

13) Modify the verbal algorithm to calculate S_n and $\$_n$ for $n = 1, 2, 3, \ldots$ when you don't know the defining function, but only know that it is an arithmetic sequence with known S_1 and d.

14) Make a similar modification for a geometric sequence with known S_1 and r.

15) Another way to derive the formula for $\$_n$ is to factor

$$\$_n = S_1 + r S_1 + r^2 S_1 + \ldots + r^{n-1} S_1 = S_1(1 + r + r^2 + \ldots + r^{n-1})$$

and substitute for $1 + r + r^2 + \ldots + r^{n-1}$ its equivalent $\frac{1 - r^n}{1 - r}$.

Show that $\frac{1 - r^n}{1 - r} = 1 + r + r^2 + \ldots + r^{n-1}$ by either multiplication or division.

16) What happens to formulas (13) and (16) when you have the constant sequence 5, 5, 5, 5, ... ? Give a formula for $\$_n$ for a constant series in terms of S_1 and n.

17) Check your second answer to exercise 16 by using the arithmetic sum formula.

2.5 Infinite Series and Limits

Two topics of central importance to advanced mathematics are those of this section, infinite series and limits. In this section we will introduce these important ideas only partially and informally, the details and more formal aspects left until you meet these topics again in your study of the calculus.

Without knowing it, you dealt with both topics as early as grade six when you tried to represent 1/3 as a decimal and found

$$(1)\quad \frac{1}{3} = .3333333333333333333333\ldots,\ ^{*}$$

always with that same tantalizing and elusive 1/3 remaining after each step of the division. We may rewrite the expression of the right member of (1) as:

$$(2)\quad \frac{1}{3} = .3 + .03 + .003 + .0003 + .00003 + \ldots$$

Notice that this right member is an infinite geometric series, infinite meaning that it continues forever or does not stop. The defining function for the sequence of terms to be added is:

$$S_n = \frac{3}{10^n}$$

You may check this by representing S_1, S_2 and S_3. In this sequence $S_1 = \frac{3}{10}$, and $r = \frac{1}{10}$. We may substitute this in one of the sum formulas of section 2.4 to get:

---

* We have been using the three dots, formally called an ellipsis, to mean "and so on following the same pattern." This is standard mathematical practice.

$$\$_n = \frac{\frac{3}{10}\left(1 - \left(\frac{1}{10}\right)^n\right)}{1 - \frac{1}{10}} = \frac{1}{3}\left(1 - \frac{1}{10^n}\right)$$

If we evaluate this expression for values of n, we get (as we should expect):

$$\$_1 = \frac{1}{3}\left(1 - \frac{1}{10}\right) = \frac{1}{3} \cdot \frac{9}{10} = \frac{3}{10} = .3$$

$$\$_2 = \frac{1}{3}\left(1 - \frac{1}{10^2}\right) = \frac{1}{3} \cdot \frac{99}{100} = \frac{33}{100} = .33$$

$$\$_3 = \frac{1}{3}\left(1 - \frac{1}{10^3}\right) = \frac{1}{3} \cdot \frac{999}{1000} = \frac{333}{1000} = .333$$

These sums check directly with what we would get in equation (2). Since we know by how the series was generated that this value gets closer and closer to 1/3 we may write formally

$$\lim_{n \to \infty} \$_n = \frac{1}{3}$$

We can read this "The limit of the series sum as n approaches infinity is one third." In our informal understanding it is reasonable for us to think of limit as meaning something we get "closer and closer to." (There are many problems with this definition, some of which will be addressed in the exercises.)

Now we consider the series formula itself as $n \to \infty$*

$$\$_n = \frac{S_1(1 - r^n)}{1 - r}$$

---

* The sense of this is "n becomes very large."

focusing our attention on r^n. We claim that

2.5.1 $$\boxed{\text{If } -1 < r < 1, \quad \lim_{n \to \infty} r^n = 0}$$

We will not prove this theorem, but will instead try to justify it by means of examples. If $r = 0.1$, it is easy to believe that r^n gets closer and closer to 0.

$$r^2 = 0.01$$

$$r^5 = 0.00001$$

$$r^{10} = 0.0000000001$$

But what about $r = 0.9$? Does $(0.9)^n$ also approach zero? Not so quickly perhaps, but it does:

$$r^2 = 0.81$$

$$r^5 = 0.59049$$

$$r^{10} = 0.3486784401$$

$$r^{50} = 0.0051537752$$

$$r^{100} = 0.0000265614$$

$$r^{200} = 0.0000000007$$

values
rounded to
ten
decimal
digits

You should check these values by calculation.

If $-1 < r < 1$, we have said that r^n approaches zero for large n. This has a profound effect on the formula for $\$_n$ under the same conditions

$$\$_n = \frac{S_1(1 - r^n)}{1 - r} \longrightarrow \frac{S_1(1 - 0)}{1 - r} = \frac{S_1}{1 - r}$$

We can write formally

2.5.2 $$\boxed{\text{If } -1 < r < 1,\ \lim_{n \to \infty} \$_n = \frac{S_1}{1 - r} \text{ for the geometric series } \$_n}$$

To apply this important theorem we notice that any geometric series that does not stop is an infinite series. We must only assure ourselves that $-1 < r < 1$. We will abbreviate $\lim_{n \to \infty} \$_n$ by $\$_\infty$ or simply $\$$.

EXAMPLE 2.5-1 Express as a rational number the decimal 0.636363...

Solution: We may rewrite .636363... as

$.63 + .0063 + .000063 + \ldots$

and identify $S_1 = .63$ and $r = .01$. Since the series is infinite and $-1 < r < 1$, we may use the formula of theorem 2.5.2 to obtain

$$\$_\infty = \frac{.63}{1 - .01} = \frac{63}{99} = \frac{7}{11}.$$

EXAMPLE 2.5.2 Find the sum of the terms of the sequence

$$\left(\frac{20}{3^n} \right)$$

Solution: We examine the first terms of this sequence

$$\frac{20}{3}, \frac{20}{9}, \frac{20}{27}, \frac{20}{81}, \ldots$$

to identify $S_1 = \frac{20}{3}$ and $r = \frac{1}{3}$.

$$\$ = \frac{\frac{20}{3}}{1 - \frac{1}{3}} = \frac{20}{2} = 10$$

In the exercises we will also examine limits of non-geometric series.

Sequences can have limits too as example 2.5-3 will show:

EXAMPLE 2.5-3. Find the limit of the sequence

$$S_n = \frac{n+6}{3-2n} \quad \text{as } n \to \infty$$

Solution: Here we examine some terms

$S_1 = 7$ $\quad S_2 = -8$ $\quad S_3 = -3$ $\quad S_4 = -2$

We seem to be going nowhere fast, but try larger values

$S_{10} = -.94^+$ $\quad S_{100} = -.538^+$ $\quad S_{1000} = -.504^-$ $\quad S_{1,000,000} = -.500004^-$

This suggests at least that $S_n \to -.5$ as a limit.

To confirm this thinking we change the form of $\left(\frac{n+6}{3-2n}\right)$ by dividing numerator and denominator by n to obtain

$$\left(\frac{1+\frac{6}{n}}{\frac{3}{n}-2}\right)$$

Now as n grows larger $\frac{6}{n}$ and $\frac{3}{n}$ will get smaller and the value of

$$\frac{n+6}{3-2n} = \frac{1+\frac{6}{n}}{\frac{3}{n}-2} \to \frac{1}{-2}$$

as we found.

Other sequences will be examined in less detail in the exercises.

Exercise Set 2.5

1 - 6 For each value of r, find by calculation r^5, r^{10}, and r^{100} correct to six decimal digits:

1) $r = .99$ (find r^{1000} also) 2) $r = -.99$

3) $r = 1.1$ 4) $r = -1.01$

5) $r = 1$ 6) $r = -1$

7) On the basis of your results in exercises 3 - 6, do you believe that the restriction of theorem 2.5.1 to $|r| < 1$ is necessary?

8) Is theorem 2.5.1 true for $r = 0$? Why?

9) Give an example of a sequence with $S_1 \neq 0$ and $r = 0$. Does the formula of theorem 2.5.2 work here?

10) Give an argument that an arithmetic sequence with $d \neq 0$ does not have a limit.

11) Even $d = 0$ does not assure a limit to an arithmetic series. Give one arithmetic series with $d = 0$ that does not have a limit and one that does.

12) Program

$$S_n = \frac{3n^2 - 10n^3 + n}{7n + 4n^3 - n^2} .$$

By evaluating S_n for increasingly large n, determine a candidate for

$$\lim_{n \to \infty} \frac{3n^2 - 10n^3 + n}{7n + 4n^3 - n^2}$$

13) Justify your candidate in exercise 12 by dividing numerator and denominator of S_n by n^3.

14) Program

$$S_n = \left(\frac{n - 20}{n + 20}\right)^5$$

(a) Evaluate S_n for $n = 1, 2, 5, 10, 15, 19, 20$.

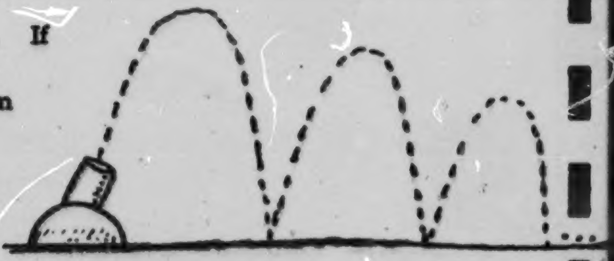
To what limit does S_n appear to be going?

(b) Now evaluate S_n for $n = 100, 1,000, 1,000,000$.

To what limit does S_n approach?

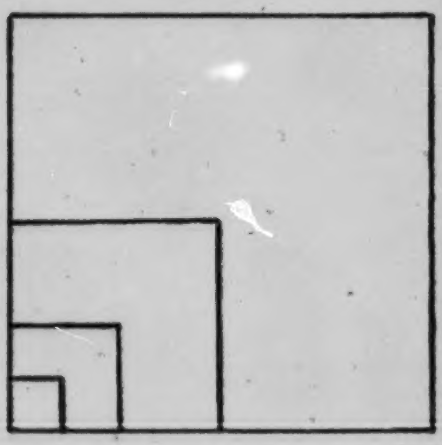
15) Justify your answer to 14(b). Hint: Use the method of Example 2.5-3.

16) A ping pong ball is shot upward from a toy cannon to a height of 5 meters. It then falls back to the table on which the cannon stands and rebounds to a height of 4 meters. On the next bounce it rises to 3.2 meters. If this sequence of bounces continues in the same pattern, how far does the ball travel before it comes to rest?



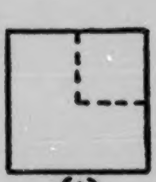
Hint: Don't forget the distance up and down!

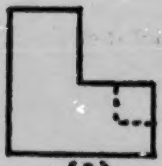
17) Starting with a square with side 8 cm., form a second square by joining the midpoints of two sides as shown. Continue this process. Find the sum of the perimeters of all squares considered separately.



18) For the nested squares of exercise 17, find the sum of the areas of all the squares considered separately.

19) From a square one fourth is removed, then one fourth of one of the remaining three fourths, and so on. If this procedure is continued, to what area does the remaining piece approach?


(1)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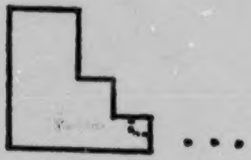
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2.6 Recursion

In this section and those following you will meet a number of extensions of the basic ideas presented in this chapter. In them when you are confronted with a new sequence or series, you will find it to your benefit to carry out the following checklist of steps. (Others may occur to you as well!)

1. A. If the sequence (or in the case of series the sequence of sums) is given by identifying S_n, write the first few terms: S_1, S_2, S_3, S_4. OR

   B. If the sequence is given as a sequence of terms, try to identify the defining function for S_n.

2. Check to see if your sequence is arithmetic or geometric by looking for a common difference (arithmetic) or common ratio (geometric). If it is one of these, be prepared to apply appropriate formulas.

3. If your sequence is neither arithmetic or geometric, examine how you can get from one step to the next. This is often useful in examining the properties of a sequence and especially in programming it.

Let us examine step 3 further by means of an example.

EXAMPLE 2.6 - 1 Find S_n for the sequence $(\frac{1}{n!})$.

Partial Solution: In case you have not met the notation n! before, we note its two equivalent mathematical definitions*

$$\boxed{\text{DEF I}\quad n! = 1\cdot 2\cdot 3\cdot\ldots\cdot n,\quad 0! = 1}$$

$$\boxed{\text{DEF II}\quad 0! = 1,\quad n! = n(n-1)!}$$

That these two definitions are equivalent may be seen by checking the results for $n = 1, 2, 3, \ldots$

	n!	
n	DEF I	DEF II
1	1	$1\cdot 0! = 1\cdot 1 = 1$
2	$1\cdot 2 = 2$	$2\cdot 1! = 2\cdot 1 = 2$
3	$1\cdot 2\cdot 3 = 6$	$3\cdot 2! = 3\cdot 2 = 6$
4	$1\cdot 2\cdot 3\cdot 4 = 24$	$4\cdot 3! = 4\cdot 6 = 24$

Now we return to our example. We seek:

$$\$_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{n!} \quad \text{(step 1 of our checklist)}$$

This series is neither arithmetic nor geometric (checklist step 2)

We can get an idea of the sum by programming and using the program to calculate $\$_n$ for increasing n. A verbal algorithm to do this is:

```
1. Set $n \leftarrow 1$, $\$ \leftarrow 0$
2. $S \leftarrow 1/n!$
3. $\$ \leftarrow \$ + S$
4. Display n, $
5. If $ is no longer changing, stop; otherwise
6. $n \leftarrow n + 1$
7. Go to step 2.
```

---

* Here as elsewhere in this chapter we consider n a natural number.

There is one difficulty with this algorithm. In step 2 we are using definition I to calculate S_n. For given n this involves n-1 multiplications (even if n! is available as a program step.) But compare S_n with S_{n+1}

$$S_n = \frac{1}{n!}$$

$$S_{n+1} = \frac{1}{(n+1)!}$$

A little thought should suggest that

$$\frac{1}{(n+1)!} = \frac{1}{n!} \cdot \frac{1}{n+1} \quad \text{(Refer to Definition II.)}$$

and

$$S_{n+1} = \frac{S_n}{n+1}$$

In other words we can go from S_n to S_{n+1} by dividing by n + 1. This leads to a modified algorithm:

```
1. Set $n \leftarrow 0$, $\$ \leftarrow 0$, $S \leftarrow 1$
2. $S \leftarrow \frac{S}{n+1}$
3. $\$ \leftarrow \$ + S$
4. Display $n + 1$, $\$$
5. If $\$$ is no longer changing, stop; otherwise
6. $n \leftarrow n+1$
7. Go to step 2.
```

You will be asked in the exercises to program and calculate this function.

What we did in Example 2.6-1 was to use a <u>recursion</u> formula $S_{n+1} = S_n/(n + 1)$ in place of the more complex defining formula $S_n = 1/n!$ A recursion formula is a formula which defines a term of a sequence by the immediately preceding term or terms.

This is possible for arithmetic and geometric sequences as well as many more complex sequences:

Arithmetic Sequence: $S_{n+1} = S_n + d$

Geometric Sequence: $S_{n+1} = r\,S_n$

The technique may be used directly for arithmetic and geometric series as well.

EXAMPLE 2.6-2 Express $\$_{n+1}$ for an arithmetic sequence in terms of $\$_n$.

Solution: $\$_n = S_1 + S_2 + S_3 + \ldots + S_n$

$$\$_{n+1} = S_1 + S_2 + S_3 + \ldots + S_n + S_{n+1} = \$_n + S_{n+1}$$

and since $S_{n+1} = S_n + d.$

$$\$_{n+1} = \$_n + S_n + d = \$_n + S_{n+1}$$

Note that in programming this would be accomplished by

$$S \leftarrow S + d$$

$$\$ \leftarrow \$ + S$$

since S_{n+1} has already been calculated in the first of the two steps. You must be very careful of such calculations.

A famous recursion defined sequence was first examined by Fibonacci of Pisa and since by many mathematicians. It is the sequence

$$S_1 = 1, \quad S_2 = 1$$

$$S_{n+2} = S_n + S_{n+1}$$

You will have a chance to examine this sequence in the exercises.

Exercise Set 2.6

1) Use the method of Example 2.6-2 to develop a recursion formula for a geometric series.

2) Translate your answer in Exercise 1 into a two step calculation algorithm.

3 - 8 To translate a sequence defining function into a recursion definition we seek k for $S_{n+1} = k\,S_n$. Clearly $k = \frac{S_{n+1}}{S_n}$. Find k for each of the following: (Note that k will usually be a function of n.)

3) (n)

4) $(2n - 1)$

5) (2^n)

6) $(2 \cdot 3^n)$

7) $(\frac{1}{n})$

8) $S_n = \frac{(n!)^2}{(2n)!}$

9 - 13 Translate into verbal algorithm steps the recursion formula for each of the following. Use your results in Exercises 3 - 8.

9) $(2n - 1)$ Solution: In Exercise 4 we found $S_{n+1} = S_n \cdot \frac{2n+1}{2n-1}$ so we have $S \leftarrow S \cdot \frac{2n+1}{2n-1}$ and then $\$ = \$ + S$.

10) (n)

11) $(2 \cdot 3^n)$

12) $(\frac{1}{n})$

13) $S_n = \frac{(n!)^2}{(2n)!}$

14) Use the second algorithm of Example 2.6-1 to program and calculate $\$_{12}$ for $S_n = \frac{1}{n!}$. Compare your answer with $e^1 - 1$, using the e^x key on your calculator.

15) Use the definition of the Fibonacci Sequence to write the first light terms of the sequence.

16) A verbal algorithm for generating terms of the Fibonacci Sequence follows. In it we will use R_n to represent storage register n.

```
1) Display 1, 1; 2, 1
        Note: This represents $n = 1, S_1 = 1; n = 2, S_2 = 1$.
2) Set $n \leftarrow 3$, $R_1 \leftarrow 1$, $R_2 \leftarrow 1$
        Note: We are using $R_1 = S_{n-1}$, $R_2 = S_{n-2}$
3) $R_3 \leftarrow R_1 + R_2$        Note: $R_3 = S_n$
4) Display n, $R_3$
5) If n is large enough, stop; otherwise
6) $n \leftarrow n + 1$, $R_2 \leftarrow R_1$, $R_1 \leftarrow R_3$
7) Go to step 3.
```

Program your calculating instrument for this algorithm and record S_n for $n = 1, 2, 3, \ldots, 12$.

17) In the case of the Fibonacci sequence the simple recursion formula replaces a complex defining function:

$$S_n = \text{INT}\left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{5}} + \frac{1}{2}\right]^*$$

Program this defining function and calculate S_n for $n = 1, 2, 3, \ldots, 12$ to compare with your answer to Exercise 16.

18 - 20 It is possible to work backwards with some sequences (but not others) to calculate S_0, S_{-1}, S_{-2}, etc. For each of the following calculate S_0, S_{-1} and S_{-2}. When possible also give a formula relating S_{-n} to S_n.

---

* Recall that INT means the number "rounded down" for numbers ≥ 0.

18) The arithmetic sequence 1, 3, 5, 7, ...

19) The geometric sequence 3, 6, 12, 24, ...

20) The Fibonacci sequence

2.7 Applications

An important application of sequences is to banking. Earlier in your study of mathematics you should have learned the basic simple interest formula

$$i = p\,r\,t$$

in which i = interest, p = principal (the amount banked or invested or loaned), r = the rate of interest (per unit of time) and t = the number of units of time.

EXAMPLE 2.7-1 Find the simple interest on a \$100 loan at 5% per year for 4 years.

Solution: Substituting in the simple interest formula:

$$i = 100\,(.05)4 = 20$$

The interest is \$20.

Compound interest differs from simple interest in that the interest for each unit of time is added to the principal so that subsequent interest is calculated on the new principal (often designated the amount).

EXAMPLE 2.7-2 Find the interest on a \$100 loan at 5% per year for 4 years compounded annually.

Solution: (by arithmetic calculation):

Year 1: $i_1 = 100\,(.05)1 = 5$

$A_1 = 100 + 5 = 105.$

Year 2: $i_2 = 105\,(.05)1 = 5.25$

$A_2 = 105 + 5.25 = 110.25$ $\quad (A_{n+1} = A_n + i_{n+1})$

Year 3: $i_3 = 110.25\,(.05)\cdot 1 = 5.5125$ *

$A_3 = 110.25 + 5.5125 = 115.7625$

Year 4: $i = 115.7625\,(.05)\,1 = 5.788125$

$A_4 = 115.7625 + 5.788125 = 121.550625$

The final amount less the principal is the compound interest so

Total interest = \$121.550625 - \$100 $\doteq$ \$21.55

That seems like a great deal of work to go through to achieve a gain of \$1.55 in interest over 4 years; however, we will see that this kind of difference is important. We now seek a short-cut to calculate compound interest.

In general, for the first unit of time, an investment p at r interest rate gives:

$$i_1 = pr \quad \text{and} \quad A_1 = p + pr = p(1+r)$$

For the second unit of time A_1 becomes the new principal and

$$i_2 = A_1 r = p(1+r)r \quad \text{and}$$

$$A_2 = A_1 + i_2 = p(1+r) + p(1+r)r^{**} = p(1+r)(1+r) = p(1+r)^2$$

Similarly for A_3, the amount after three units of time:

$$A_3 = p\,(1+r)^3$$

And for A_4: $A_4 = p(1+r)^4$

---

* Note: Some readers will be tempted to round answers at each step. Practice here varies but in many cases today banks only round at the end of a transaction.

** Be sure that you see how this factoring is accomplished.

This pattern provides the basic

(2.7-1)

COMPOUND INTEREST FORMULA
$A_n = p(1 + r)^n$

in which A_n is the amount accumulated after n time units on principal, p, at r rate of interest per time unit.

EXAMPLE 2.7-3 Find the interest on a \$100 loan at 5% per year for 4 years compounded annually. (This is Example 2.7-2 again.)

Solution: Substituting in the compound interest formula (2.7-1):

$$A_4 = 100\,(1.05)^4 = 121.550625$$

This amount less the original principal represents the interest earned: \$121.55 - \$100 = \$21.55

In the formula $A = p(1 + r)^n$, the role of each factor, $1 + r$, may be considered as contributing two separate parts to the amount:

$$1 + r \begin{cases} \text{the multiplier 1 gives back the amount from the beginning of the period} \\ \text{the multiplier } r \text{ gives the interest for the period} \end{cases}$$

This is then applied for n interest periods to give the effect $(1 + r)^n$.

In the past interest was often compounded annually, semi-annually, quarterly, or monthly. Today, with access to computational tools, interest is almost always compounded daily. Here is a table that shows how stated interest rates may be converted to effective annual rates:

stated interest rate	rate for interest period	multiplier per interest period	multiplier on amount per year	effective annual interest rate
r compounded annually	r	$1 + r$	$1 + r$	r
r compounded semi-annually	$\frac{r}{2}$	$1 + \frac{r}{2}$	$(1 + \frac{r}{2})^2$	$(1 + \frac{r}{2})^2 - 1$
r compounded quarterly	$\frac{r}{4}$	$1 + \frac{r}{4}$	$(1 + \frac{r}{4})^4$	$(1 + \frac{r}{4})^4 - 1$
r compounded monthly	$\frac{r}{12}$	$1 + \frac{r}{12}$	$(1 + \frac{r}{12})^{12}$	$(1 + \frac{r}{12})^{12} - 1$
r compounded daily	$\frac{r}{360}$	$1 + \frac{r}{360}$	$(1 + \frac{r}{360})^{365*}$	$(1 + \frac{r}{360})^{365} - 1$

EXAMPLE 2.7-4 Find the interest on $10,000 for 35 days at 5% per year compounded daily.

Solution: If the annual interest rate is 5% the daily rate is $\frac{.05}{360} = .00013888\ldots$ Using this value accurate to the limit of our calculator, we have:

---

* Notice the different numbers here. Today banks are in competition to offer the highest effective annual interest rate, but they are restricted by law to a maximum r. (One reason for this legal restriction is that home mortgage and loan rates must go up as interest rates rise.) To make the effective rate as large as possible the divisor, 360, is used as the number of days in a so-called "banker's year." The 365 then represents the number of days in the calendar year. Advertising often refers to this difference as "five additional interest days."

$$A = 10,000\,(1.000138889)^{20}$$

$$= 10027.81446 \text{ which rounds to } \$10027.81$$

$$i = \$10,027.81 - \$10,000 = \$27.81$$

EXAMPLE 2.7-5 Calculate the daily multiplier and the effective annual interest rate for the following annual rates compounded daily: 5%, $7\frac{3}{4}$%, 9%

Solution: You should check the following calculations:

Rate	Daily Multiplier	Effective Annual Rate
r	$1 + \frac{r}{360}$	$(1 + \frac{r}{360})^{365} - 1$
.05	1.000138	.05199769
.0775	1.00021527	.08173683
.09	1.00025	.09553036

The key to compound interest is to be found in the factors

$$(1 + r)^n$$

We now examine a more general application of such factors. To lead to this, suppose for a moment that we had 100% interest per year. Such a large interest rate - which might suggest loan sharks - would return \$2.00 for \$1.00 after a year. Greedily we seek to increase our return still further by compounding. This would result in calculating

$$(1 + \frac{1}{n})^n$$

for n interest periods. (We do not allow the extra days of our daily interest calculating in Exercise 2.7-5.) Check the following calculations of this function for increasing values of n:

n	$(1 + \frac{1}{n})^n$	
1	2	
2	2.25	(semi-annually)
4	2.4414+	(quarterly)
12	2.613+	(monthly)
52	2.692+	(weekly)
365	2.714+	(daily)
365.24	2.718+	(hourly)
365.24.60	2.71826+	(by the minute)

It turns out that

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n = 2.718281828+^{*}$$

and this important mathematical constant is assigned the name $\underline{e}$. In fact we may define e^x as

$$e^x = \lim_{n\to\infty} (1 + \frac{x}{n})^n$$

There is an e^x key on your calculator. You should check to see that e^1 is indeed 2.71828+. You will meet this ubiquitous function again in the exercises and in Chapter 6. (In fact you already met it in exercise 2.6-14.)

*

We turn our attention now to a quite different problem, a gambling problem that is translated into a series exercise.** We supply the probability background only superficially.

---

* One way to remember the decimal approximation to this constant is to think of it as 2.7 followed by two 1828's: thus, 2.7 1828 1828+. This repetition does not continue: e is like π in this regard.

** For this problem we are indebted to Stephan L. Snover and Mark A. Spikell.

EXAMPLE 2.7-6 A carnival booth offers the following game. You toss a fair coin (chance of heads 1/2, chance of tails 1/2) until you toss the first head. As soon as that happens you are to be paid in dollars the square of the number of tosses. What would be a fair (break-even) price to pay to play this game?

Solution: Here is a chart showing the number of tosses against the amount you win

TOSS FOR FIRST HEAD	1	2	3	4	...	n
PAYOFF	1	4	9	16	...	n^2

Thus, for example, if heads came up first on the fourth toss you would win $16. Simple probability theory shows that your chances are:

TOSS FOR FIRST HEAD	1	2	3	4	...	n
CHANCE	1/2	1/4	1/8	1/16		1/2n

and your expected gain would be the sum of the series

$$\$ = \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 9 + \frac{1}{16} \cdot 16 + \dots + \frac{n^2}{2^n} + \dots$$

For the game to be fair this is also the amount you should pay to play. Since $ is neither an arithmetic nor a geometric series, a reasonable solution is by calculation via a program. The procedures of section 2.5 applied here should suggest the limit.

$$\lim_{n \to \infty} \$_n = 6$$

Be sure that you carry out this calculation. (See exercise 2.7-1). This result means that \$6 is a fair price to pay to play this game. If you're asked to pay \$10 to play you will lose on the average \$4 per game.

Exercise Set 2.7

1) Program and calculate the value of \$ for example 2.7-6.

2) Try the game for yourself ten times. Find the average payoff per game. Do your results agree with the calculated value? (You may wish to combine values with those of your classmates to obtain an average of a larger number of plays.)

3) Modify your program in exercise (1) to calculate the fair cost of the game if the payoff is the cube of the number of tosses to first head.

4) Test the game of exercise (3) by ten trials.

5 - 8. Find the effective annual interest rates for the following loans:

5) 6% compounded quarterly

6) 10% compounded monthly

7) $8\frac{1}{2}$% compounded daily

8) $5\frac{3}{4}$% compounded daily

9) Credit card purchases are often charged $1\frac{1}{2}$% or 2% interest on the unpaid balance each month. Thus the multiplier per month is 1.015 or 1.02. Find the effective annual rate of this charge. Comment!

10 - 16. Loan and mortgage payments become complicated by repayments of principal. Thus the balance or principal on which interest is calculated is changing all the time. Here you can work through a simplified example to

see how this works. You take out a loan of $1000 at 10% interest per year compounded annually. We seek the amount of the equal payments if you are to pay at the end of each year for ten years.

10) If there were no interest how much would you pay each year?

11) If you paid $100 each year on the principal and paid the interest for that year your payments would be unequal. Calculate them. Partial solution: Year 1: $100 + .10 ($1000); year 2: $100 + .10 ($900); etc.

12) Find the total of these payments.

13) How much interest is paid?

14) Compare this with the interest on $500* at 10% per year compounded annually (with no repayments).

15) Find the average monthly payment in exercise (11). For equal repayments this may be the monthly payment.

16) For the first five years, who suffers by the method of exercise (15)?

17) Peter Minuit in 1626 purchased Manhattan Island from the Indians for trinkets worth about $24. That seems like a remarkable deal, given the current value of the island real estate. But let's examine this value. Suppose instead of investing in real estate the Dutch colonist had stayed in Rotterdam and invested carefully in securities paying an effective annual rate of 7%. What would his investment be worth today?

18) Develop a program to calculate the amount to which Minuit's investment would have accrued for a given input year. Use it to determine

---

* Starting with a loan of $1000 and ending with a loan of $0 means that the average loan was $500.

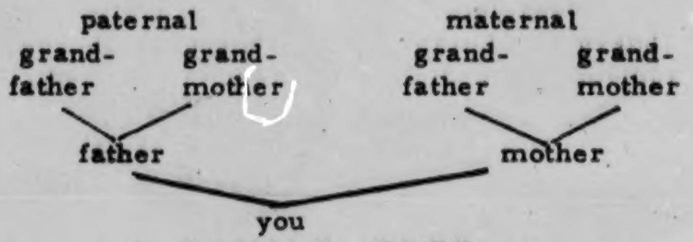
the value in 1700, 1776, 1800, 1864, 1900, 1918, and 1970.

19- 27 Sometimes calculations can be misleading as this series of questions will show. We will calculate the approximate number of ancestors you had at the time of Julius Caesar about 2040 years ago.

19) How many (biological) parents do you have?

20) How many grandparents do (or did) you have?

21) How many great grandparents did you have?



A FAMILY TREE

22) This pattern suggests that the number of relations is multiplied by what number each generation?

23) If one generation is 30 years, how many generations are there in 2040 years?

24) Using your answers in 22 and 23, calculate the approximate number of relatives you had 2040 years ago in Julius Caesar's time.

25) Calculate by this means how many ancestors you would have had at the time of Homer about another 480 years earlier.

26) What is wrong with these results? In answering you may wish to consult an encyclopaedia, almanac, or history book to find the approximate world population today and 2040 years ago.

27) Where did our calculations go astray?

2.8 Chapter 2 Test

(1-4) Identify each of the following sequences as (a) arithmetic (b) geometric (c) neither arithmetic or geometric.

1) $3, 3/2, 3/4, 3/8, \ldots$
2) $5, -10, 20, -40, \ldots$
3) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
4) $6, 2, -2, -6, \ldots$

5) Give an example of a sequence that is both arithmetic and geometric.

(6-7) Find the tenth term of the sequence $12, -6, 3, -\frac{3}{2}, \ldots$

6) as a decimal
7) as a common fraction

(8-11) Find the indicated term of each of the following sequence or series.

8) $\{\tan\theta, 1, \text{ctn } \theta, \ldots\}$, S_{30}
9) $\{1, -\frac{1}{5}, \frac{1}{25}, \ldots\}$, $\$_{8}$
10) $\{20, 14, 8, \ldots\}$ S_{17}
11) $\{1, 3, 5, 7, \ldots\}$ $\$_{20}$

(12-13) Find the sum of these finite series.

12) $500 + 250 + 125 + \ldots + 3.90625$
13) $1 + 2 + 3 + \ldots + n$

(14-15) Find the "sum" of these infinite series.

14) $1 + .9 + .09 + .009 + \ldots$

15) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$

16) Express $2.435\overline{435}$ as a rational expression in simplest form.

17) Find the limit of the sequence $S_n = \frac{n+3}{2-3n}$ as $n \to \infty$.

18) Find the limit of the sequence

$$S_n = (1 + \frac{1}{n})^n \text{ as } n \to \infty .$$

19) Determine the exact value of 14!

20) Given the geometric sequence $S_1 = 10$ and $r = .1$, find d for the arithmetic sequence formed by taking the log of each term.

CHAPTER 3. ITERATION, MATHEMATICAL INDUCTION, AND THE BINOMIAL THEOREM

In Chapter 3, we turn our attention to some fundamental mathematical techniques, concepts, and notation. Your calculating power and more specifically your ability to program a calculator or computer to do routine tasks should give you the tools not only to solve problems but also to understand the underlying concepts here.

3.1 Over and over and over and... Iteration

You have almost certainly met problems before that appeared to be extremely difficult but that turned out to be transparently simple once you were able to develop the right approach.* We consider one such problem here:

$$\text{Simplify } \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}$$

This example certainly looks difficult. Most of us are already uncomfortable with radicals - they turn up those messy irrationals like $\sqrt{2} = 1.41421356\ldots$ - and here we have still worse: radicals within radicals. Not only that but these square roots and twos go on forever. Given this example on a test our first response might well be to move on to the next question. That would be unfortunate, because you will see that the problem is quite reasonable and may in fact be solved in several ways.

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*This does not imply that there are no truly hard problems, but mathematics by its very nature often leads to remarkable and unexpected simplifications.

EXAMPLE 3.1-1 Simplify $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}$

Solution 1: Mathematicians (and especially textbook writers) do not usually assign impossible exercises, so we expect that there is indeed a value to this expression. Can we get an idea at least of what that value is by calculation? Indeed we can.

Think of the expression as a kind of sequence:

$\sqrt{2}$, $\sqrt{2+\sqrt{2}}$, $\sqrt{2+\sqrt{2+\sqrt{2}}}$, and so on.

We can calculate these values quite easily by working from the innermost radical in each case. Check the following with your own calculator:

$$\sqrt{2} = 1.4142^{+}$$
$$\sqrt{2+\sqrt{2}}\ ^{*} = 1.8478^{-}$$
$$\sqrt{2+\sqrt{2+\sqrt{2}}} = 1.9616^{-}$$
$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} = 1.9904^{-}$$

Now it is not necessary to start over to carry out each of these calculations. Notice that you get each subsequent value by adding 2 and taking the square root of the sum. These steps are:

AH: $\boxed{+}\quad\boxed{2}\quad\boxed{=}\quad\boxed{\sqrt{\ }}$

RPN: $\boxed{2}\quad\boxed{+}\quad\boxed{\sqrt{\ }}$

Continue this procedure until your result no longer changes. To save effort and to avoid errors it is a good idea to program these steps.

---

*On algebraic calculators especially don't forget to complete the addition before taking the second square root. The steps here could be $\boxed{2}\quad\boxed{\sqrt{\ }}\quad\boxed{+}\quad\boxed{2}\quad\underset{\equiv}{\boxed{=}}\quad\boxed{\sqrt{\ }}$

The procedure you followed in Solution 1 is called an iterative procedure. To iterate means quite simply to repeat. Perhaps our most important example of an iterative process comes from biology: breathing. There are many, many others in our world.

Iteration is very closely related to recursion which we studied in section 2.6. In fact you may consider the two words synonyms. Here if we assign values to our approximations we can easily develop a recursion formula for our solution.

Let:

$$x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}}$$

and let

$$x_1 = \sqrt{2}\,, \quad x_1 \text{ the first approximation}$$

$$x_2 = \sqrt{2 + \sqrt{2}}\,, \quad x_2 \text{ the second approximation}$$

$$x_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}\,, \quad x_3 \text{ the third approximation}$$

$$\vdots$$

From what we found in Solution 1, we can write the recursion formula:

$$x_{n+1} = \sqrt{2 + x_n}$$

which, together with $x_1 = \sqrt{2}$, will generate the sequence of approximations we found in Solution 1.

So far what we have done only formalizes Solution 1. But there is a tremendous bonus in this formalization as we will see. In fact Solution 1 is really not even a solution in a strict sense. We merely found that the calculator value no longer changed. If you think about it for a minute or two you will realize that a calculator with more decimal places would have allowed continuing change. The calculator "solution" then just led us to a very good guess. Let's see then what Solution 2 will give us.

EXAMPLE 3.1-1 (again): Simplify $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}$

Solution 2: We identified the iteration formula

$$x_{n+1} = \sqrt{2 + x_n}$$

on page 3.1 - 3. Now we ask ourselves what would happen when, in our iteration or recursion or repetition, we truly no longer changed from one step to the next? At that time we should expect

$x_{n+1} = x_n = x$, the value of the complex radical.

If that is the case, we need only substitute x for x_{n+1} and x_n to get

$$x = \sqrt{2 + x}$$

This is an easy equation to solve by the following steps

$x^2 = 2 + x$ (by squaring)

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2, -1$

Since $x = -1$ is extraneous (why?) we have the expected solution $x = 2$.

You will meet other iterative processes in the exercises and then will apply this technique to the solution of equations in the next section.

Exercise Set 3.1

1) Use the method of Solution 1 (page 3.1 - 2) to find

   $y = \sqrt{6+\sqrt{6+\sqrt{6+\ldots}}}$.

2) What is the recursion formula for the iteration in exercise 1?

3) Use your recursion formula of exercise (2) and the method of Solution 2 (page 3.1 - 4) to find y in exercise (1).

(4 - 11) Another setting in which iteration occurs is in the study of continued fractions. We will not study this interesting topic in detail in this course but will consider only specific cases of infinite continued fractions, that is fractions of the form

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \ldots}}}$$

Continued fractions have what are technically called convergents, that is values that approach or converge to their value. For the example above the convergents are:

$$a, \quad a + \frac{1}{b}, \quad a + \cfrac{1}{b + \cfrac{1}{c}}, \quad a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d}}}, \quad \ldots$$

In the following exercises use these ideas to find the value of x in:

(a) $$x = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}$$

4) Write the first four convergents for (a). Leave these answers in fractional form. Label these convergents x_1, x_2, x_3, and x_4.

5) In your answer to (4), locate the part of x_4 that is the same as x_3. Draw a loop around this part.

6) Similarly loop the part of x_3 that is the same as x_2; and following the same pattern loop the part of x_2 that is the same as x_1.

7) If you have performed exercise (6) carefully, your work should justify $x_1 = 2$, $x_2 = 2 + \frac{1}{x_1}$. Continue this pattern for $x_3 = f(x_2)$ and $x_4 = f(x_3)$.

8) Generalizing from (7), express x_{n+1} as a function of x_n.

9) Program your calculator or computer to compute x_{n+1} from x_n following your recursion formula for (8).

10) Starting with $x_1 = 2$, calculate successive convergents until $x_{n+1} = x_n$. Give the value of x_n when this happens.

11) Replace x_{n+1} and x_n by x in your answer to (8). Solve for x.* How does this answer compare to your result in (10)?

(12 - 20) Another of the many applications of iteration techniques is to the calculation of roots. If we wish to compute $\sqrt{N}$ by iteration one approach is by the following means:

Let $x = \sqrt{N}$

$x^2 = N$ (squaring)

$x = \frac{N}{x}$ (dividing by x)

---

*You will need to recall that for $ax^2 + bx + c = 0$, $x = (-b \pm \sqrt{b^2 - 4ac})/2a$.

$$2x = x + \frac{N}{x} \quad \text{(adding x)(*)}$$

$$x = \frac{1}{2}\left[x + \frac{N}{x}\right] \quad \text{(dividing by 2)}$$

We now set

$$\text{(b)} \qquad x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

and we have our recursion formula for finding $\sqrt{N}$.

12) Program your calculator to compute x_{n+1} from x_n in formula (b). Assume that N is stored in R_0, and if necessary store x_n in R_1. List the program steps.

13) Use your program in exercise (12) to compute $\sqrt{19}$. Store 19 in R_0. Enter 4 in the display (as x_1). Give your answer after x_n no longer changes. Compare this result with $\sqrt{19}$ calculated by use of the $\boxed{\sqrt{\ }}$ key.

14) Why was 4 a good choice for x_1?

15) Repeat exercise (13) but using $x_1 = 1$.

16) Repeat exercise (13) but using $x_1 = 1000$.

17) Repeat exercise (13) but using $x_1 = -10$. How does this solution differ from those of exercises 13 - 16? Is it correct?

18) Use the same method as the one before exercise (12) to derive a recursion equation for $\sqrt[3]{N}$. (Hint: At (*) add 2x.)

19) Program your calculator for the iteration of exercise (18) and use it to compute $\sqrt[3]{50}$. Use $x_1 = 1$. Check your answer by cubing it by means of the $\boxed{y^x}$ key.

20) Develop a formula for computing $\sqrt[5]{N}$, program it and use this

program to compute $\sqrt[5]{1000}$. (Hint: At (*) add 4x.)

It may be of interest to you that the program of exercise (12) is often the one that is preprogrammed (or "hardwired") into calculators for the $\boxed{\sqrt{\;}}$ key. These hardwired programs usually use either $x_1 = 1$ or $x_1 = N$. The iterative technique of exercises (12) - (20) is named after the famous English mathematician and scientist, Sir Isaac Newton. (You will recall perhaps that Newton is supposed to have thought of the concept of gravity when he saw an apple fall from a tree.) The technique is variously called Newton's Method or the Newton-Raphson Method. In its more general form it is a calculus technique.

3.2 Solving Equations by Iteration

In exercises 12-20 of Section 3.1 you have already used iteration techniques to solve equations of the form $x^n = N$. In this section we will first solve systems of equations by iteration and then return to the solution of single equations.

Figure 3.2-1 shows the graph of the system of equations

$$\begin{cases} x + y = 5 & \text{(line } \ell \text{)} \\ y = 2x - 3 & \text{(line m)} \end{cases}$$

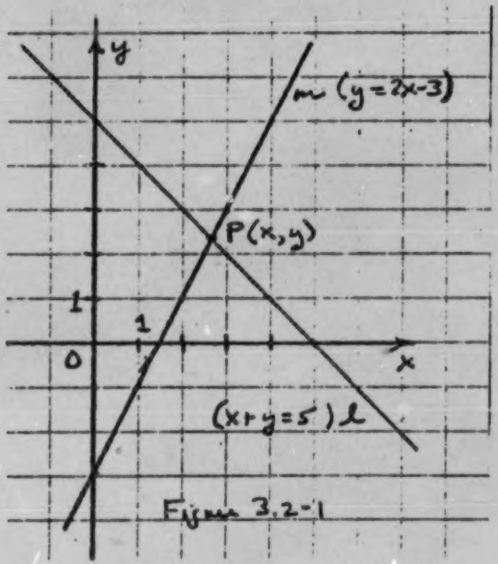


Figure 3.2-1

We know, of course, that the solution of this system of equations is found at point P of the graph. But here we want to use the graph instead to justify our iteration procedure.

We convert the equations of lines ℓ and m to the following:

$$\begin{cases} y = 5 - x & \text{(line } \ell \text{)} \\ x = \dfrac{y + 3}{2} & \text{(line m)} \end{cases}$$

Be sure to satisfy yourself that the algebra is correct. Notice that one equation is solved for y in terms of x, the other for x in terms of y.

We start with an arbitrary choice for x, $x_1 = 0$. Substituting this into the equation for line ℓ gives $y_1 = 5$. Substituting 5 for y in the equation for line m gives $x_2 = 4$. Continuing to trade back and forth between the two equations, always substituting the current value of x or y to obtain a new value for the other, we obtain

n	x_n		y_n
1	0	$\xrightarrow{\ell}$	5
		$\overset{m}{\swarrow}$	
2	4	$\xrightarrow{\ell}$	1
		$\overset{m}{\swarrow}$	
3	2	$\xrightarrow{\ell}$	3
		$\overset{m}{\swarrow}$	
4	3	$\xrightarrow{\ell}$	2
		$\overset{m}{\swarrow}$	
5	2.5	$\xrightarrow{\ell}$	2.5
		$\overset{m}{\swarrow}$	
6	2.75	$\xrightarrow{\ell}$	2.25
		$\overset{m}{\swarrow}$	
7	2.625	$\xrightarrow{\ell}$	2.375

The arrows show how each value contributes to the next.

Table 3.2-1

What is going on here? What is happening is that we are slowly converging on the solutions x and y of this system of equations. We can see this on the graph. $x_1 = 0$ is a vertical line, in fact the y-axis. When we substitute this x value into the equation for line ℓ, the y value that results ($y_1 = 5$) is the y value where these two lines intersect as shown in Figure 3.2-2.

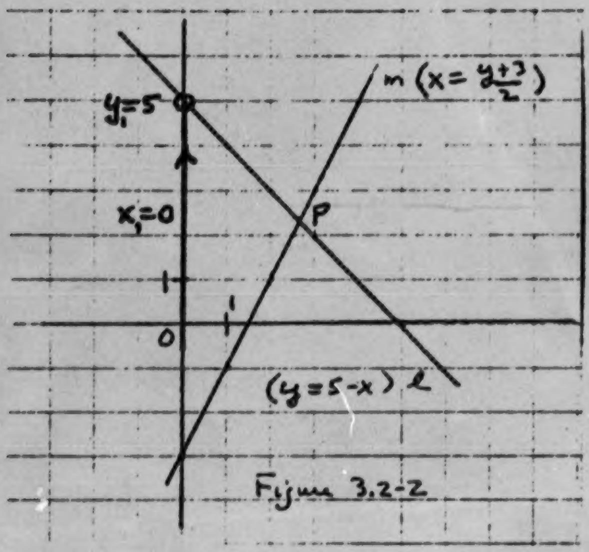

$m\ (x = \frac{y+3}{2})$
$y_1 = 5$
$x_1 = 0$
P
1
1
0
$(y = 5 - x)\ l$

Figure 3.2-2

Now y = 5 is a horizontal line. When we substitute this value into the equation for line m, the resulting x value ($x_2 = 4$) is the x coordinate of the intersection of y = 5 with m. This part of our trail is marked in Figure 3.2-3.

We have only to continue this process to see how Table 3.2-1 is formed. Additional segments of our spider web trail toward P are shown on Figure 3.2-4.

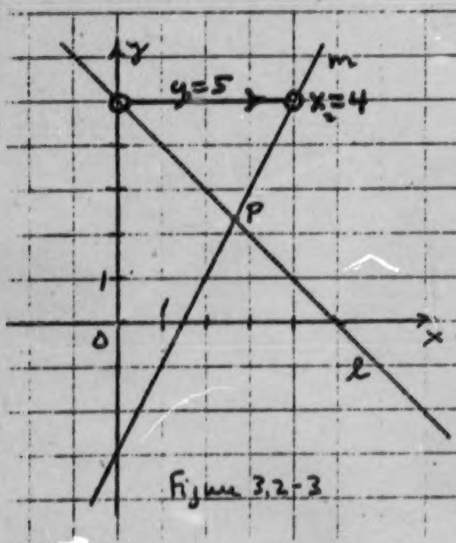

y
m
y = 5
$x_2 = 4$
P
1
1
0
x
l

Figure 3.2-3

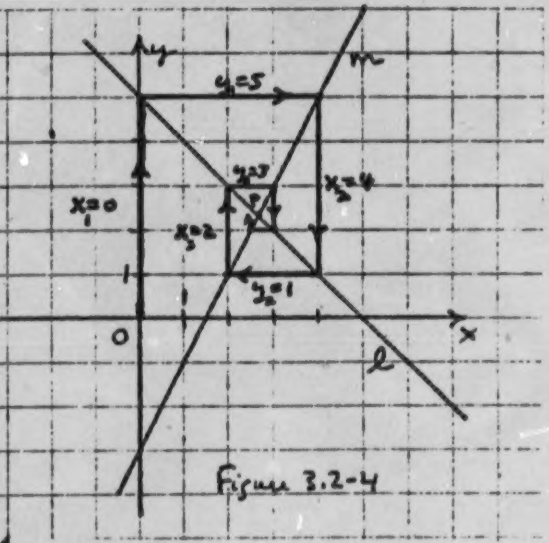

y
m
$y_1 = 5$
$y_3 = 3$
$x_2 = 4$
$x_1 = 0$
P
$x_3 = 2$
1
$y_2 = 1$
1
0
x
l

Figure 3.2-4

The argument we have presented and Figure 3.2-4 should justify our claim that the values of x_n and y_n in Table 3.2-1 are converging on (x,y), the solution of this system of equations.

But wait a minute. Why should we go to all this work when we have much more straightforward methods of finding solutions to systems of equations? The answer to this question is important. There are many more complex equations that also respond to the iteration technique that we have used when none of the other methods work. We have developed and justified the technique with a simple example. Now let us apply it to a more difficult problem.

EXAMPLE 3.2-1 Solve for y: $y = 2^{-y^2}$.

Solution: $y = 2^{-y^2}$ is a complex equation indeed. One possible way to solve it is by trial and error. Choose a value of y, substitute it, simplify, and see if the equation balances. For example, if $y = 1$

$$2^{-y^2} = 2^{-(1)^2} = 2^{-1} = \tfrac{1}{2}$$

but since the left side of the equation is 1, we are not too close.

Another route to a solution is to use iteration. To do this we introduce another variable, x, to give us a second equation. You should convince yourself that:

$$y = 2^{-y^2} \text{ is equivalent to } \begin{cases} y = 2^{-x} & (1) \\ x = y^2 & (2) \end{cases}$$

The system of equations may be solved by iteration. Again we start with a guess, $x_1 = 0$, substitute it in equation (1) to find y_1, substitute this y-value into equation (2) to get x_2 and so on. You should program your calculator to check our progress in Table 3.2-2* (see exercise 4):

n	x_n		y_n
1	0	$\xrightarrow{(1)}$	1
		$\xleftarrow{(2)}$	
2	1	$\xrightarrow{(1)}$	0.5
		$\xleftarrow{(2)}$	
3	0.25	$\xrightarrow{(1)}$	0.84^+
		$\xleftarrow{(2)}$	
4	0.71^-	$\xrightarrow{(1)}$	0.61^+
⋮	⋮		⋮
25	0.500^-		0.707^+
26	0.500^+		0.707^+

Table 3.2-2

Our original equation did not call for x, so we claim

$$y \doteq 0.707$$

---

*An algorithm to accomplish this is:

```
1) Set a = 0.                                  (This is $x_1$)
2) $a \leftarrow 2^{-a}$, display a.           (This generates $y_n$)
3) Stop if enough pairs have been generated.
4) $a \leftarrow a^2$, display a.             (This generates $x_n$)
5) Go back to step 2.
```

You should check this in the original equation to see that it balances (approximately).

To show how our iteration process relates to the graphs of these functions, Figure 3.2-5 displays the first several steps:

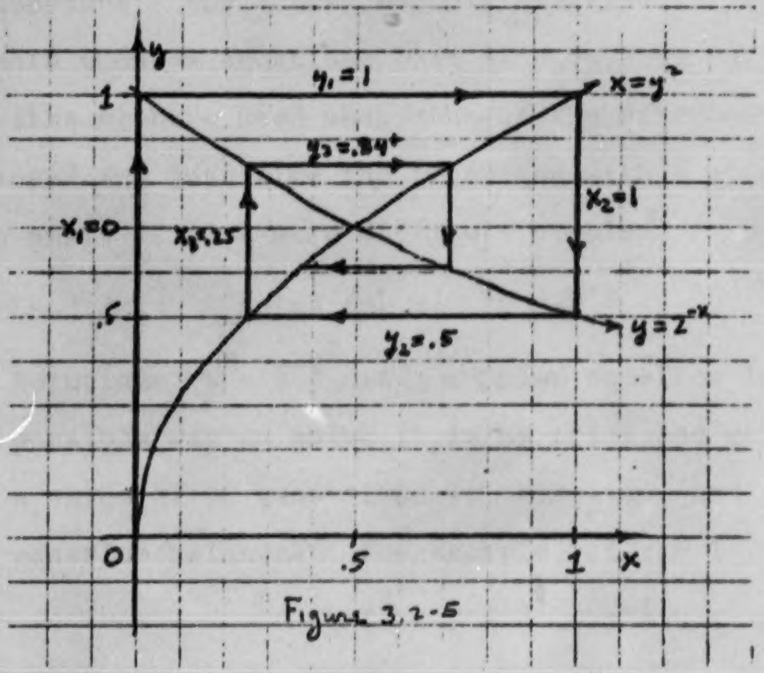


Figure 3.2-5

You need not be concerned at this time how the graphs themselves were constructed.

In the exercises you will explore these techniques further and use them to solve other problems.

Exercise Set 3.2

1) Program your calculator to carry out the iteration for the system of equations on page 3.2 - 1. An algorithm for this process is:

```
1. Set $a = 0$                          (This is $x_1$)
2. Let $a \leftarrow 5-a$, display a    (This displays $y_n$)
3. Let $a \leftarrow (a+3)/2$, display a    (This displays $x_n$)
4. Stop when enough pairs have been generated
5. Go back to step 2.
```

Use this program to calculate x_n and y_n until the two values (separately) remain unchanged to 3 decimal places. (You should check your first few values against Table 3.2-1.

2) Compare your answers in (1) to the graph in Figure 3.1-1. Do your answers seem reasonable?

3) Solve this same system of equations $\begin{cases} x + y = 5 \\ y = 2x - 3 \end{cases}$

by another algebraic method. How does this answer compare with your iteration solution of exercise (1)?

4) Program your calculator to carry out the iteration of Example 3.2-1. The algorithm for this is given on page 3.2 - 5. Use your program to find x_6 and y_6 to 2 decimal digit accuracy. (You may want to build a counter into your program, but it is enough here to count steps yourself.)

5) In the iteration of Example 3.2-1, to what value does x appear to converge? (See Table 3.2-1) Use this value and the equation $y = 2^{-x}$ to express y in radical form.

6) How does your answer in (5) for y compare to y_{26} in Table 3.2-2?

7) Does your answer in (5) check in the original equation $y = 2^{-y^2}$?

(8 - 14) In these exercises we seek an iteration solution for

$$\begin{cases} 3y - 4x = -3 & (1) \\ 3y - 2x = 3 & (2) \end{cases}$$

8) Solve equation (1) for x in terms of y.

9) Solve equation (2) for y in terms of x.

10) Develop an algorithm for generating successive approximations x_n and y_n to this system.

11) Program your algorithm of exercise (10) and use it to converge on values of x and y. What are these values?

12) Figure 3.2-6 is a graph of the system you have solved. How do your answers in (11) compare with the coordinates of P on the graph?

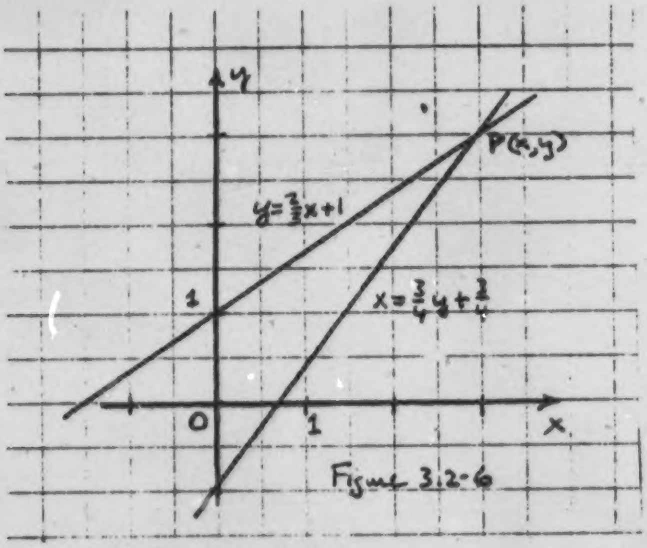


Figure 3.2-6

13) Copy Figure 3.2-6 on coordinate paper and sketch the first few steps of your iteration ($x_1 = 0$, $y_1 = 1$, $x_2 = 1.5$, ...). Why do you suppose that this type of iteration is called a stepladder?

14) Solve the system of equations given before exercise (8) by another algebraic means and compare with your answer.

(15 - 22) These exercises are designed to expose some problems associated with iteration techniques.

15) Recall the system of equations of page 3.2 - 1:

$$\begin{cases} x + y = 5 & (\ell) \\ y = 2x - 3 & (m) \end{cases}$$

We solved equation (ℓ) for y and equation (m) for x. Surely it would have been easier to leave equation (m) solved for y and solve equation (ℓ) for x:

$$\begin{cases} x = 5 - y & (\ell') \\ y = 2x - 3 & (m) \end{cases}$$

But what happens now when we start with x = 0 and iterate between equations (m) and (ℓ)? In a table of the form of Table 3.2-1, give values for x_n and y_n through n = 4.

16) We say that the values in exercise (15) are diverging. Why? (It will be helpful if you look up the words diverge and converge in the dictionary.)

17) Suppose we had a system of equations which we converted to the system

$$\begin{cases} x = f(y) & (1) \\ y = g(x) & (2) \end{cases}$$

only to find that iteration of these equations diverged. What does your answer to exercise (15) suggest would be a reasonable procedure to convert your system to one which would converge?

18) Solve the system of equations for x and y by iteration:

$$\begin{cases} y = 4 - 2x \\ x = y - 1 \end{cases}$$

19) Try to solve the following system by iteration:

$$\begin{cases} 2x + y = 3 \\ 2x - y = 1 \end{cases}$$

What is the relation between x_{n+2} and x_n? Between y_{n+2} and y_n?

20) Copy the graph of Figure 3.2-7 on coordinate paper and trace the iteration route starting with $x_1 = 0$. What happens?

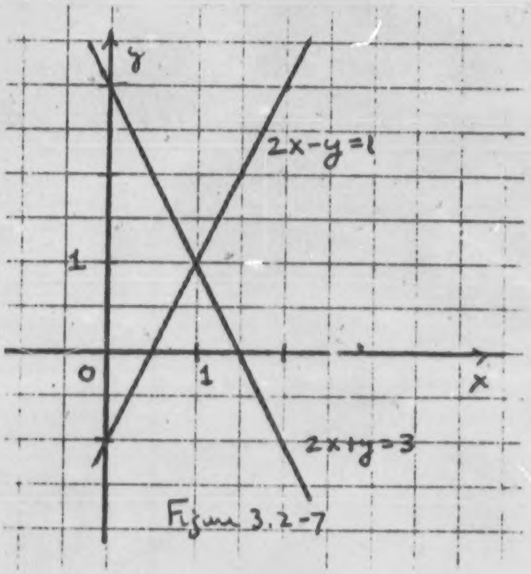


Figure 3.2-7

21) Does it matter which equation is solved for x and which for y?

22) If the slope of one linear equation is the negative of the slope of the other, this iteration technique does not work. What are the slopes of the lines in exercise (19)?

23) Solve the following system of equations by iteration starting with $x_1 = 0.5$.

$$\begin{cases} y = x^2 - 1 & (1) \\ x = \dfrac{y + 2}{4} & (2) \end{cases}$$

24) Suppose in exercise (23) we were only interested in the value of x. We could combine the two recursion equations into one by the following steps (copy and complete):

$$x = \frac{(\qquad) + 2}{4}$$ substituting from equation (1) into equation (2)

$$x = \frac{(\quad) + 1}{4}$$ simplifying

25) To iterate from x_n to x_{n+1} we could use this last equation with subscripts:

$$x_{n+1} = \ ?$$

26) Solve the equation in exercise (24) by programming the iteration in exercise (25).

27) Rewrite the equation in exercise (24) as a quadratic equation equal to zero.

28) Check your solution in exercise (26) in the equation of exercise (27).

3.3 Some Applications of Iteration

Consider a thin rectangular metal plate ABCD. Sides $\overline{AB}$ and $\overline{CD}$ are kept at 0° C by melting ice packed against them. Sides $\overline{AD}$ and $\overline{BC}$ are kept at 100° C by boiling water. (These temperatures cannot be maintained at the corners, of course, but this will not affect our problem.) Clearly the plate will vary in temperature from point to point. We seek the equilibrium temperatures at points P_1 and P_2. See Figure 3.3 - 1.[*]

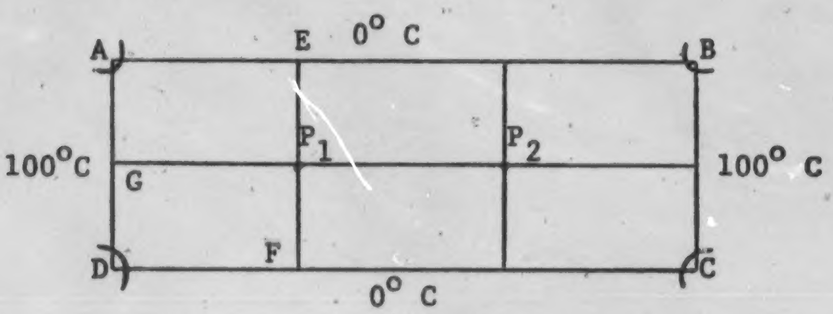


Figure 3.3 - 1

To solve this problem we employ a thermodynamic principle: The temperature at any point in the metal sheet is the average of the temperatures at "neighboring" points. Thus, for example, the temperature at P_1 is the average of the temperatures at E, P_2, F, and G. Thus, if we represent the temperatures at P_1 and P_2 by t and T, we have:

$$\begin{cases} t = \dfrac{0 + T + 0 + 100}{4} \\ T = \dfrac{0 + 100 + 0 + t}{4} \end{cases}$$

---

[*] This and some other problems of this section were communicated to us by P. Rosenbloom and S. Schuster.

or in simplified form as:

$$t = \frac{T + 100}{4}$$

$$T = \frac{t + 100}{4}$$

EXAMPLE 3.3-1. Solve by iteration techniques the system of equations:

$$t = \frac{T + 100}{4}$$

$$T = \frac{t + 100}{4}$$

Solution: We need only guess a starting value, say $t_1 = 50$ to initiate the iteration:

Table 3.3 - 1

n	t_n	T_n
1	50	37.5
2	34.37^+	33.59^+
3	33.39^+	33.34^+
4	33.337^+	33.334^+
5	33.333^+	33.333^+

Continuing with this iteration suggests that temperatures at P_1 and P_2 are each $33\frac{1}{3}\ {}^\circ C$.

EXAMPLE 3.3-2 Solve the system of equations in Example 3.3-1 using $t_1 = 0$.

Solution: This time the iteration would be

Table 3.3 - 2

n	t_n	T_n
1	0	25
2	31.25	32.81^+
3	33.20^+	33.30^+
4	33.32^+	33.33^+
5	33.333^-	33.333^+

You should notice two things about Examples 3.3-1 and 3.3-2. First, the iteration converges to the same value. Second, the iteration in Example 3.3-2 is only slightly slower in converging.

We have "solved" the problem posed on page 3.3 - 1 only crudely. In example 3.3-3 we will show the direction we would go in refining this kind of problem.

EXAMPLE 3.3-3 Find equilibrium temperatures at P_1, P_2, ..., P

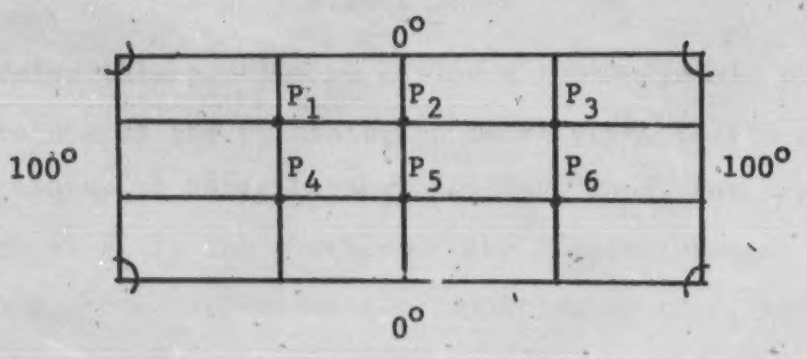


Figure 3.3-2

Solution: Let the temperature at P_i be t_i (for $i = 1, 2, \ldots, 6$)

Then we have:

$$t_1 = \frac{0 + t_2 + t_4 + 100}{4} = \frac{t_2 + t_4 + 100}{4}$$

$$t_2 = \frac{0 + t_3 + t_5 + t_1}{4} = \frac{t_1 + t_3 + t_5}{4}$$

$$t_3 = \frac{0 + 100 + t_6 + t_2}{4} = \frac{t_2 + t_6 + 100}{4}$$

$$t_4 = \frac{t_1 + t_5 + 0 + 100}{4} = \frac{t_1 + t_5 + 100}{4}$$

$$t_5 = \frac{t_2 + t_6 + 0 + t_4}{4} = \frac{t_2 + t_4 + t_6}{4}$$

$$t_6 = \frac{t_3 + 100 + 0 + t_5}{4} = \frac{t_3 + t_5 + 100}{4}$$

Now we can set up our chart for determining these six values. We have arbitrarily chosen initial value of all $t_i = 50$

n	$t_{1,n}$[*]	$t_{2,n}$	$t_{3,n}$	$t_{4,n}$	$t_{5,n}$	$t_{6,n}$
1	50	50	50	50	50	50
2						
3						
4						

Table 3.3-3

To fill in this table use the iteration equations and the last value of the appropriate t's. Thus $t_{1,2} = \frac{50 + 50 + 100}{4} = 50$. $t_{2,2} = \frac{50 + 50 + 50}{4} = 38$, etc. You will be asked in Exercise (1) to extend this table.

---

[*] The first subscript number designates the point which the temperature represents, the second the iteration step. You need not worry about this notation.

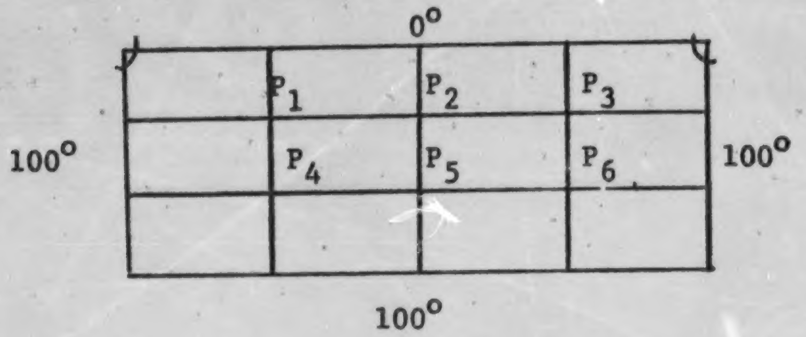
Exercise Set 3.3

1) Continue the iteration in Example 3.3-3 to $n = 4$. Use direct (not programmed) calculation and round answers to whole degrees to simplify computation.

2) What regularities do you find in your values in exercise (1)? Justify these regularities by reference to Figure 3.3-2.

3) We could have simplified computation in Example 3.3-3 a great deal by taking advantage of the symmetries of Figure 3.3-2. Let t be the temperature at points P_1, P_3, P_4, and P_6 and T be the temperature at P_2 and P_5. Develop two (instead of six) iteration equations and find t and T to the nearest degree. Do your answers check with exercise (1)?

4) Use iteration techniques to find equilibrium temperatures at P_1, P_2, ..., P_6 on Figure 3.3-3.



Be sure to notice that only one side is 0°.

Figure 3.3-3

5) Use iteration techniques to find equilibrium temperatures at P_1, P_2, and P_3 on Figure 3.3-4.

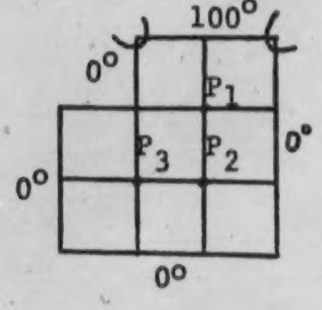


Figure 3.3-3

(6 - 10) Maze or labyrinth problems are quite similar to the physics problems we have just studied. In what follows we will treat probability ideas very informally (but correctly!). Consider the maze of Figure 3.3-5. Suppose that this

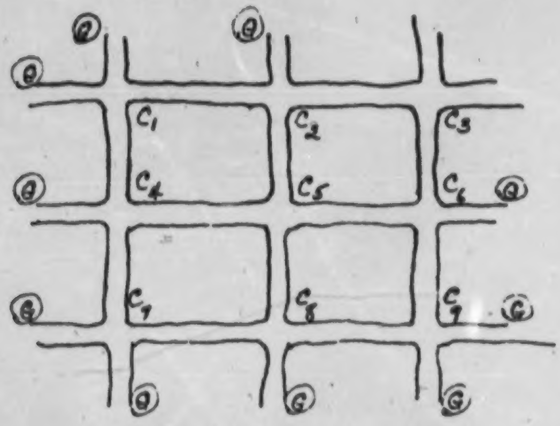


Figure 3.3-5

maze is made up of a series of tunnels meeting at C_1, C_2, ..., C_9, and with the open ended tunnels representing exits. A prisoner wanders through this maze randomly. Each time he comes to a corner (one of the C's) he rests a moment and then goes on, picking any of the four directions quite by chance. (They all look alike to him, even the direction from which he came.)

Of course it is possible that the prisoner will wander forever. The chance of this is in fact so small we eliminate it. We assume that sooner or later he will emerge from one of the open exits. All but two of the exits are guarded by guards (as marked Ⓖ but the two exits from C_3 are left unguarded. We want to know the chance of the prisoner escaping from each of the corners.

To solve this problem we make the following assumptions:

(1) Chance of escape at a guarded exit: 0

(2) Chance of escape at an unguarded exit: 1

(3) Chance of escape from a corner C_i: $0 \le P_i \le 1$[*]

(4) The chance of escape at any corner is the average of the chances at adjacent corners and exits.

(For example $p_1 = \frac{0 + p_2 + p_4 + 0}{4}$.)

6) Write the recursion equations for the probabilities of escape from corners $C_2 - C_9$.

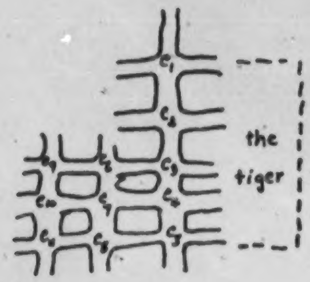
7) Use your equations in exercise (7) to determine by iteration
9 the values of $p_1, p_2, \ldots, p_6$, accurate to two decimal places. (Use any initial values of each p_1, $0 \le p \le 1$.)

8) In your answer to exercise (7) which is the best starting point? the worst? Do these answers conform to your expectations in looking at Figure 3.3-5?

---

[*] It should be clear to those at all familiar with probability that in these circumstances $0 < P < 1$. In fact assumption (3) is not even necessary to this problem.

9) Worse days ahead. Another maze. With apologies to Frank Stockton, a lady lost, a tiger guarding one bank of exits - chance of escape there 0. We seek her chance for escape from each corner of this maze. (Her choices are random as in Exercises 6 - 8)



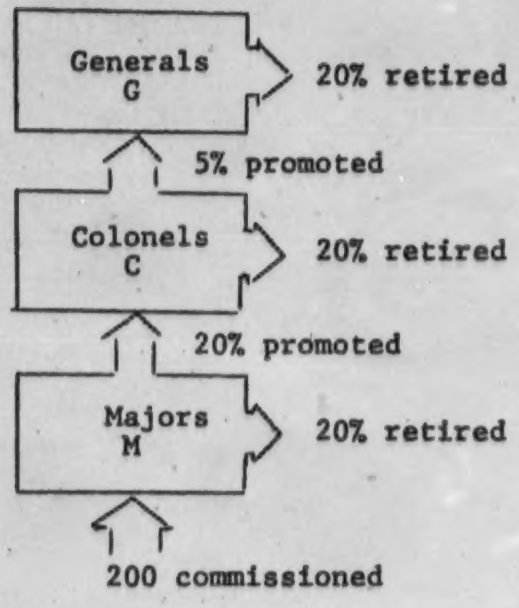
Note: There is another approach to this kind of problem that students with computer or microprocessor access may wish to explore. This is the method of simulation. The computer simulates the condition of the problem and then carries out a great many trials. (Hundreds of ladies are sacrificed in the process - theoretically.) Then the computer assigns as probability of escape from a given corner the number of escapes from that corner divided by the number of trials from that corner.

10) Make up your own maze with new conditions (for example more or fewer branches at corners) but no more than six exits.

11)* A small country has an army officer corps of 1000 officers. A new promotional system is inaugurated with the following annual conditions:

---

* This exercise is due to A. Engel.


Generals
G
20% retired
5% promoted
Colonels
C
20% retired
20% promoted
Majors
M
20% retired
200 commissioned

This promotion system may be represented by means of three replacement statements. For example, for generals we have:

$$G \leftarrow \underset{\substack{\uparrow \\ \text{because 20\% retire}}}{.8G} + \underset{\substack{\uparrow \\ \text{promotions}}}{.05C}$$

State the replacement statements for colonels, C, and majors, M. (Don't forget to subtract both promoted and retired.)

12) Choose an initial number of officers of the three ranks subject to the condition G + C + M = 1000. Make a table to represent the annual army status. Compute numbers of officers for n = 1 (your initial values) to n = 20. (Round numbers to integers. Why?)

n	G	C	M
1			
2			
⋮			

13) Demonstrate that the initial condition of 1000 officers is not necessary by repeating the iteration of exercise (12) for

(a) G = C = M = 0, or

(b) G = 2000, C = 0, M = 0. (In this country the president was elected on the promise that he would appoint all his supporters generals.)

14) When these systems reach equilibrium so that the number of generals, colonels, and majors no longer change from year to year, the "←" in your replacement statements of exercise (11) will be replaced by =. Make this change in your three equations, solve them algebraically and compare the answers with your answers to exercises (12) and (13).

3.4 Undetermined Coefficients

You have seen by now that calculators and computers are great at spewing out long lists of numbers. All you have had to do is program some recursion function with a loop and you have a machine like Figure 3.4-1:

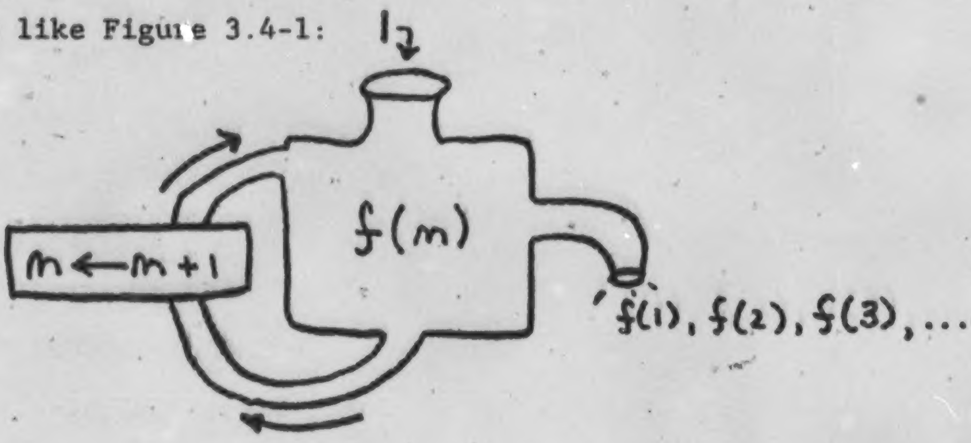


Figure 3.4-1

A simple example of this process is the sequence (2n-1). It is processed as in Figure 3.4-2.

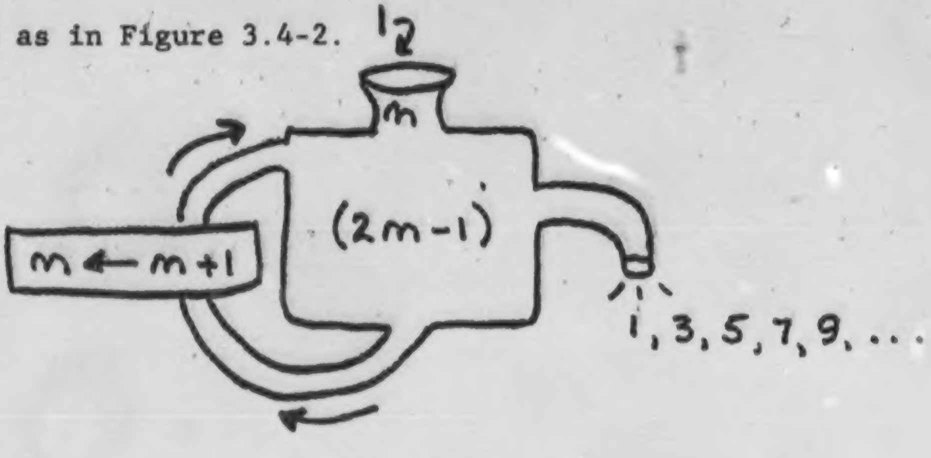


Figure 3.4-2

Without a signal to stop such a machine will continue to spew out odd numbers ...327,329,331,..., 1000083,1000085,1000087, ... until your computing instrument breaks down.

So we have the quite reasonable (and only slightly limited[*]) idea:

FUNCTIONS LEAD TO NUMBERS

Here, in fairness, we explore the converse notion:

NUMBERS LEAD TO FUNCTIONS

and we set out to develop some means of carrying this out. Using the analogy of our same function machine, we have the situation of Figure 3.4-3:

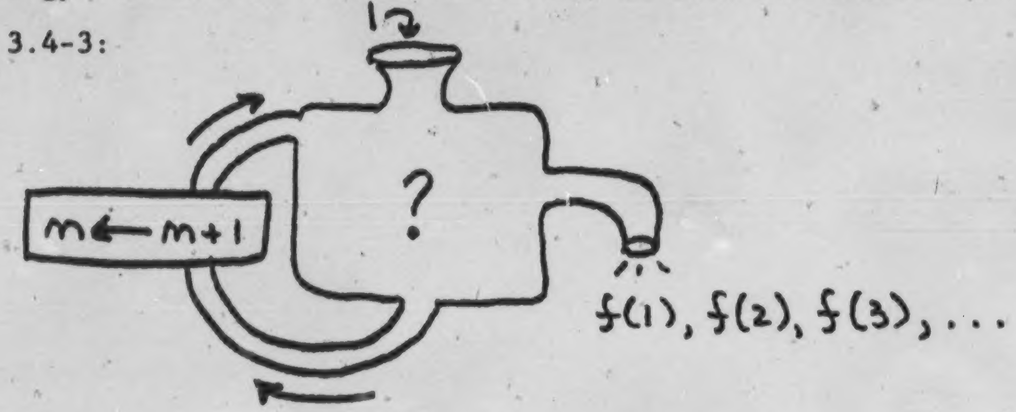


Figure 3.4-3.

This is not just a theoretical situation. You can see clearly what is meant by the following two person exercise. Each of you should program the following algorithm. (Do not let your partner see the program steps!)

---

[*] Not limited in this text, but limited more generally by the fact that there are mathematical functions on many things that are not numbers, geometric shapes for example.

```
1.  Let n = 1
2.  a ← f(n), display a
        (Here f is any function you choose.  It can be as
         simple as 2n or as complex as, say, $\ell n \sqrt{e^n - \tan^{-1}} n/\sin$
3.  When you have accumulated enough values, stop
4.  n ← n+1
5.  Go back to step 2.
```

Now set your calculating device to run (or clear your display on a microprocessor) and trade calculators. Each of you will have a machine that will generate numbers. Your challenge is to decide - without looking - what is the function of step 2 of the algorithm. Unless the function is quite simple, you're probably stumped.

Be assured that this is far more than a little problem designed to make you work in the mathematics classroom. Much scientific activity is of exactly the form described here. The scientist collects data and looks for regularities in that data. And the best kind of regularity is a functional equation that will allow him to predict the future - or unknown parts of the past.

In most of the rest of this section we restrict ourselves to polynomial sequences; in exercise (16), however, we'll hint at how the idea is extended. To deal with polynomial sequences we apply the following rule:

> WHEN Δ^n ARE FIRST CONSTANT, THE POLYNOMIAL FUNCTION IS OF DEGREE n

Isn't that a wonderful rule? Well, we'll have to admit, no, not yet. We first have to show what it means. First we'll show you (or remind you) what we mean by the degree of a polynomial.

POLYNOMIAL[*]	DEGREE	
$ax + b$	1	(linear)
$ax^2 + bx + c$	2	(quadratic)
$ax^3 + bx^2 + cx + d$	3	(cubic)
	⋮	
$ax^n + bx^{n-1} + \ldots + q$	n	(n-th degree)

Simple enough? It should be. The degree of a polynomial is the highest power of the variable. Now let us see how we determine what that degree is.

EXAMPLE 3.4-1. Determine the degree of the polynomial function f, for the following data.

Table 3.4-1

n	f(n)
1	0
2	4
3	20
4	54
5	112
6	200

---

[*] Here a, b, c, etc., represent constants like 6 or π. They are called parameters. For a specific quadratic for example, we might have a = 1, b = 2, and c = 5. The function would then be $x^2 + 2x + 5$. But since $ax^2 + bx + c$ is a general statement, a, b, and c can vary from function to function. This leads to the contractory seeming (and upsetting) idea of a parameter as a variable constant!

Solution: We determine the differences $(\Delta^n)^*$ for f(n)

n	f(n)	Δ^1	Δ^2	Δ^3
1	0			
		4		
			12	
2	4			6
		16		
			18	
3	20			6
		34		
			24	
4	54			6
		58		
			30	
5	112			
		88		
6	200			

Notice that these differences are found merely by subtracting values in the preceding column. Since Δ^3 are the first differences that are constant (they are each 6), f(n) is of degree 3.

Our solution of Example 3.4-1 tells us that this data leads to an equation of the form:

$$f(n) = an^3 + bn^2 + cn + d$$

Our task in deciding what specific function f is, now reduced to finding the values of a, b, c, and d. These are the undetermined coefficients of this method. We will see how to complete this

---

* Δ is the Greek letter delta. It is often used in mathematics to represent a difference or change. For example slope is often defined

$$m = \frac{\Delta y}{\Delta x}$$

task in Example 3.4-2.

EXAMPLE 3.4-2. Find the function, f, for Example 3.4-1.

Solution: We know from Example 3.4-1 that we seek a, b, c, and d in

(1) $f(x) = an^3 + bn^2 + cn + d$

We can find them by using our original data in Table 3.4-1. For example we know that when $n=1$, $f(n) = 0$. Substituting this into (1) we have:

(2) $0 = a + b + c + d$

When $n = 2$, $f(n) = 4$. Substituting these values in (1):

(3) $4 = 8a + 4b + 2c + d$

When $n = 3$, $f(n) = 20$:

(4) $20 = 27a + 9b + 3c + d$

When $n = 4$, $f(n) = 54$:

(5) $54 = 64a + 16b + 4c + d$

Now we can stop. We have four equations in four unknowns and we can solve them for a, b, c, and d. Solving them is messy but not hard:

First eliminate d:

Equation numbers

(6) = (3) - (2) : $4 = 7a + 3b + c$

(7) = (4) - (3): $16 = 19a + 5b + c$

(8) = (5) - (4): $34 = 37a + 7b + c$

Then eliminate c by the same process on these new equations:

(9) = (7) - (6): $12 = 12a + 2b$

(10) = (8) - (7): $18 = 18a + 2b$

and b from these

(11) = (10) - (9): $6 = 6a$

and this giving, finally:

$$a = 1.$$

Substitute this into (9) gives:

$$12 = 12(1) + 2b$$

which leads to

$$b = 0.$$

Using these values in equation (6) gives:

$$4 = 7(1) + 3(0) + c$$

which leads to

$$c = -3$$

And finally, these then values in (2) yields:

$$0 = 1 + 0 - 3 + d$$

or

$$d = 2.$$

Thus we find that our undetermined coefficients are $a = 1$, $b = 0$, $c = -3$, and $d = 2$. We plug them into our function to give

$$f(n) = (1)n^3 + (0)n^2 + (-3)n + (2)$$

or

$$f(n) = n^3 - 3n + 2$$

We have already said that our method of undetermined coefficients is limited to polynomial equations. Two other limitations are of concern here:

(1) We must have enough terms to generate the needed differences to determine degree. Thus in Example 3.4-1 we needed at least five data pairs. (Why?)

(2) We assume that the pattern continues.

Exercise Set 3.4

(1 - 4) Determine the degree of the functions in each exercise:

1)

n	f(n)
0	4
1	3
2	6
3	13
4	24
5	39

2)

n	g(n)
0	1
1	4
2	7
3	10
4	13
5	16

3)

n	h(n)
0	3
1	2
2	9
3	30
4	71
5	138

4)

n	j(n)
0	5
1	14
2	21
3	26
4	29
5	30

5) Find f(n) in Exercise (1).

6) Find g(n) in Exercise (2).

7) Find h(n) in Exercise (3).

8) Find j(n) in Exercise (4).

9) How does the inclusion of the pair (0, f(0)) simplify exercises (5) - (8)?

10) Find $f(n)$ for the data:

n	$f(n)$
0	0
1	-12
2	-18
3	-18
4	-12
5	0

11) Find $f(n)$ for the data:

n	$f(n)$
0	10
1	8
2	6
3	4
4	2
5	0

(12 - 15) These exercises are designed to justify in part the method we have been using without proof. We will consider the case for a quadratic function

$$f(n) = an^2 + bn + c$$

12) Copy and complete the table

n	$f(n)$ $an^2 + bn + c$
0	c
1	$a + b + c$
2	
3	
4	

13) Complete, in terms of a, b, and c, Δ^1 and Δ^2 columns for your function in exercise (12).

14) What is Δ^2 in exercise (13)? Does this confirm our rule for determining degree for a quadratic?

15) Confirm the degree law of page 3.4 - 3 for a cubic, that is

$$f(x) = an^3 + bn^2 + cn + d$$

by the methods of exercises (12) - (14). You will of course need a Δ^3 column.

(16 - 20) In these exercises we show one direction taken for non-polynomial functions.

16) Of course many functions are not polynomial functions. For example, consider the function $f(n) = 2^n$

n	f(n)
0	1
1	2
2	4
3	8
4	16
5	32

Complete Δ^1, Δ^2, Δ^3, and Δ^4 columns. What is happening?

17) To determine equations from data like that of exercise (16) - assuming that the function is not known as it is here - mathematicians must look to other techniques. The rapid growth of f(n) suggests the possibility of an exponential function here and one way to "tame" exponential functions is to convert them to polynomial functions by taking logs. Here if we substitute (in the exercise (16) data) log f(n) we have

n	log f(n)
0	0
1	.301
2	.602
3	.
4	
5	

Copy and complete the table to 3 decimal digit accuracy.

18) Determine the degree of the polynomial relating n to log f(n) by forming the Δ^n columns until they are constant.

19) Let y = log f(n). Set y equal to the polynomial in n whose degree is identified in exercise (18). Use the data of exercise (17) to evaluate coefficients of the polynomial.

20) Replace y in your answer to exercise (19) by log f(n). Your result should now be of the form

$$\log f(n) = \text{polynomial in } n$$

Write each side as a power of ten:

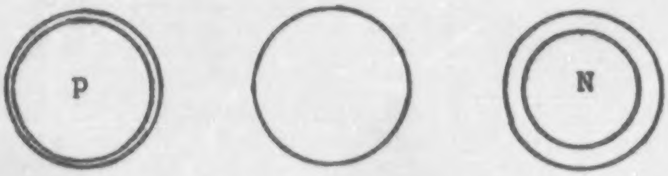
$$10^{\log f(n)} = 10^{\text{polynomial in } n}$$

Show that your answer simplifies to the expected result

$$y = 2^n$$

(21 - 24) These exercises show how the method of undertermined coefficients may be applied to real data.

21) Draw three circles on a sheet of scrap paper and place a penny , p, and a nickel, N, in the end circles as shown.



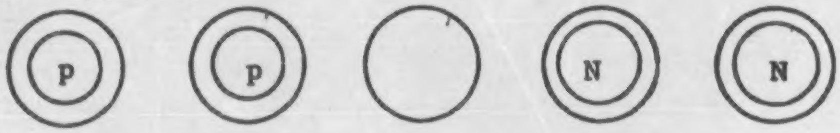
We want to know the minimum number of moves necessary to reverse the positions of the nickel and the penny according to the following two allowable moves:

(1) a coin may be moved to an adjacent circle if it is empty

(2) a coin may jump over one coin of the opposite type if the circle beyond it is empty.

How many moves does this task require?

22) Now we extend the game to two coins of each type and five circles.



By the same rules, what is the minimum number of moves that it takes now to exchange positions? (Be careful that you don't get yourself blocked.)

23) Extend the game to three coins of each type and seven circles, and then four coins of each type and nine circles. Make a table to record your data.

n	f(n)
1	
2	
3	
4	

n - number of coins of each type

f(n) - minimum number of moves to complete the reversal of positions

24) Find the function f in your data of exercise (23).

25) Challenge your neighbor to a calculator duel. Each of you program a polynomial function of degree no higher than 3, and with no more than 2 terms into your calculator. Use the algorithm of page 3.4 - 2. Exchange calculators and see who can determine the function first. This must be done by using the data provided in run mode. It is not allowed to examine program steps.

3.5 Mathematical Induction

Mathematical (or finite) induction is an important mathematical method of proof which is often confused with other similar terms, in particular scientific induction and deduction. For that reason we will focus on mathematical induction by comparing it with these related terms.

In science induction and deduction represent two contrasting approaches to problems.

DEDUCTION: General laws apply to specific cases

INDUCTION: Specific cases lead to general laws.

Now deduction is a process used in mathematics without difficulty. This method was the basis of the proofs you developed in geometry. But induction as described here is not a strong enough method for mathematical proof. The problem is that we cannot be sure that we have taken into account all cases. We will show you this by means of an example we met in another context earlier in this chapter.

EXAMPLE 3.5-1 Prove by (scientific) induction that

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}$$

for any finite number of twos is always less than 2.

Solution attempt: We check the truth of our statement for various numbers of twos. (Confirm these values by calculation.)

One 2: $\sqrt{2} = 1.5^-$ less than 2

Two 2's: $\sqrt{2+\sqrt{2}} = 1.9^-$ less than 2

Three 2's: $\sqrt{2+\sqrt{2+\sqrt{2}}} = 1.97^-$ less than 2

Four 2's: $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}} = 1.991^-$ less than 2

Five 2's: $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}} = 1.998^-$ less than 2

Six 2's: $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}} = 1.9994^-$ less than 2

Seven 2's: $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}} = 1.999^-$

.
.
. less than 2

Fifteen 2's: 1.999999998^- less than 2

Here we're faced with two insurmountable problems. First, our calculator soon doesn't have enough accuracy and even suggests that the theorem is untrue. For my calculator (ten digit accuracy) for example:

Seventeen 2's: 2.0 ?

The second problem is that even with an "infinitely" accurate calculator, we could never list all cases. Thus mathematical proof by scientific induction fails.

We have seen that scientific induction cannot be a mathematical technique. Before we leave scientific induction, however, we should point out that it is still an extremely important technique in science. What it lacks is only the surety we seek in mathematics. This final lack is the reason why we hear of so many scientific theories, whose strength is supported by data collected over centuries.

Still there exists the possibility of disproving one of these theories by displaying a single exception. An example of how this works may be seen in Newton's Theory of Gravitation which was accepted by scientists for several hundred years until Einstein found exceptions that required modifications in that theory.

Let us return to our example now to see how we can prove it by the method of mathematical induction. We will then formalize that process.

EXAMPLE 3.5-2 Prove by mathematical induction that

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}}$$

for any finite number of two's is less than 2.

Solution: We start as in Example 3.5-1:

One 2: $\sqrt{2} = 1.5^-$ Less than 2

But now we recall the recursion procedure we used to get from one number of 2's to the next:

$x_{n+1} = \sqrt{2 + x_n}$, where n is the number of two's so far.

We argue that

$$x_k < 2$$

$$2 + x_k < 4 \quad \text{(adding 2 to each member)}$$

$$\sqrt{2 + x_k} < 2 \quad \text{(taking positive square ro}$$

$$x_{k+1} < 2 \quad \text{(since } s_{n+1} = \sqrt{2 + x_n}\text{)}$$

Thus we see that whenever $x_k < 2$, then $x_{k+1} < 2$. Using this fact and the fact we established earlier, that $x_1 < 2$, we have

$x_1 < 2$

k = 1 $x_2 < 2$ (using $x_k < 2$ implies $x_{k+1} <$

k = 2 $x_3 < 2$ " " " "

k = 3 $x_4 < 2$ " " " "

.

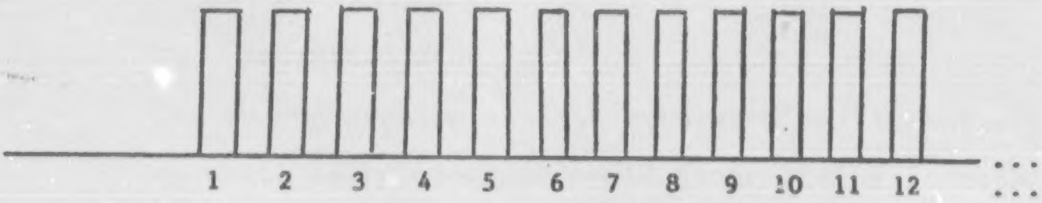
.

.

k = n $x_n < 2$

This process will continue and we have proved that the radical expression is indeed always less than 2.

Let us apply the process of mathematical induction to a simple and familiar physical model. We line up dominoes stood on end in such a way that when one domino falls so too does the next.



Now tip the first domino against the second. Doing so starts an inductive process: 1 tips 2, 2 tips 3, 3 tips 4, ..., n tips n+1,.... No matter how many dominoes are lined up in this way they will ALL fall! Notice the two parts of what we did to tip all dominoes:

(1) We tipped the first domino

(2) We had the dominoes set up so that each domino tipped would tip the next.

Stated formally we have justified

THE PRINCIPLE OF FINITE INDUCTION (PFI)

For any statement $S(n)$ about n, an integer $\geq i$, when

(1) $S(i)$ is true, and

(2) whenever $S(k)$ is true, then $S(k+1)$ is true (k an integer $\geq i$)

then $S(n)$ is true for all $n \geq i$.

In this principle i is most often taken to be equal to one and the principle then applies to the natural numbers.

In Example 3.5-2 we established Part (1) of the principle by showing $x_1 = 1.5^- < 2$. In that same example we established Part (2) of the principle by showing that $x_k < 2$ implied $x_{k+1} < 2$. Now we'll use the principle in a new example.

EXAMPLE 3.5-3. Find a formula for

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)}$$ and prove

it by the Principle of Mathematical Induction (PFI).

Solution: First we seek a formula by examining S_n for $n = 1, 2, 3, 4$

$$S_1 = \frac{1}{1\cdot 2} = \frac{1}{2}$$

$$S_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$S_3 = S_2 + \frac{1}{3\cdot 4} = \frac{2}{3} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$$

$$S_4 = S_3 + \frac{1}{4\cdot 5} = \frac{3}{4} + \frac{1}{20} = \frac{16}{20} = \frac{4}{5}$$

A reasonable guess for a formula for S_n is $S_n = \frac{n}{n+1}$.

Now we set out to prove that this formula works. Part (1) of PFI. We must show S_1 of the series ($S_1 = \frac{1}{1\cdot 2}$) is equal to S_1 of our guessed formula ($S_1 = \frac{1}{1+1}$). Since both equal $\frac{1}{2}$, we have established Part (1).

Finally we must prove Part (2) of PFI. Given $S_k = \frac{k}{k+1}$ we must show that $S_{k+1} = \frac{k+1}{(k+1)+1}$ by recourse to S_k and to our series definition. To do this we proceed as follows:

$S_k = \frac{k}{k+1}$ Hypothesis of Part (2) of PFI

$S_{k+1} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$ by the series definitions for n = k+1.

$$= \frac{k(k+2) + 1}{(k+1)(k+2)} = \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1}$$

Notice how in proving Part II of PFI, we are given that our formula is true for n = k. (This is like saying that we are given in this part that the k^{th} domino will fall.) We then use this hypothesis to prove that we can get the formula for $n = k+1$ by recourse to our given information, in this case the series definition.

The Principle of Mathematical Induction is a most important one. When Guiseppe Peano reduced the number of postulates about number systems to five, this principle was one of the five! In fact it provides the method of proof that is basic to development of Peano's system beyond his five axioms. You will have an opportunity to apply this important principle in the exercises.

Exercise Set 3.5

(1 - 5) In these exercises you are given a statement about a natural number, n. By trying a few values of n, decide which statements you think are true and which false. Do not try to prove any of them. (Your calculator should help.)

1) $2n^2 + 1$ is a prime number.

2) $2^n > n^3$.

3) $2 \cdot 4^n + 1$ is divisible by 3.

4) $n^2 + n + 41$ is a prime number, given that the following numbers are among the primes: 43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223, 251, 281, 313, 347, 383, 421, 461, 503, 547, 593, 641, 691, 743, 797, 853, 911, 971, 1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601.

5) The sum of the first n odd numbers is n^2.

6) The number of chords joining n points on a circle is $n(n-1)/2$.

7) In exercise (2) find a natural number value of n for which the statement is true. Do you think that this statement is true for all values of n larger than this?

(8 - 12) We seek to prove the statement of exercise (5): The sum of the first n odd numbers is n^2.

8) True or false: The n^{th} odd number may be represented $2n-1$.

9) The following is equivalent to the statement we wish to prove: $1 + 3 + 5 + \ldots + (2n-1) = \underline{\ ?\ }$

10) Prove Part (1) of PFI for your completed statement in (9).

11) Prove Part (2) of PFI for your completed statement in (9), that is

$$\text{Given: } 1 + 3 + 5 + \ldots (2k-1) = k^2$$
$$\text{Prove: } 1 + 3 + 5 + \ldots + \underline{[2(k+1)-1]} = (k+1)^2$$

(Hint: add the underscored term to both sides of the given equation.)

12) Do your proofs in (10) and (11) establish the truth of the theorem $1 + 3 + 5 + \ldots + (2n-1) = n^2$ for all n?

(13 - 16) We seek to prove statement (6): The number of chords joining n points on a circle is $n(n-1)/2$.

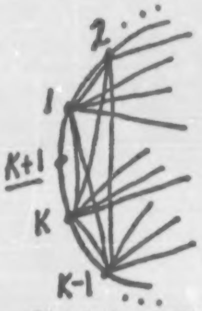
13) Prove Part (1) of PFI for this statement, for $n = 1$. (How many chords are there? Is this the same value as that of the formula?

14) To prove Part (2) of PFI for this statement, recall that we have

Given: Joining k points are k(k-1)/2 chords.

What value do we want to establish for k+1 points? (Hint: Substitute k+1 in the formula.)

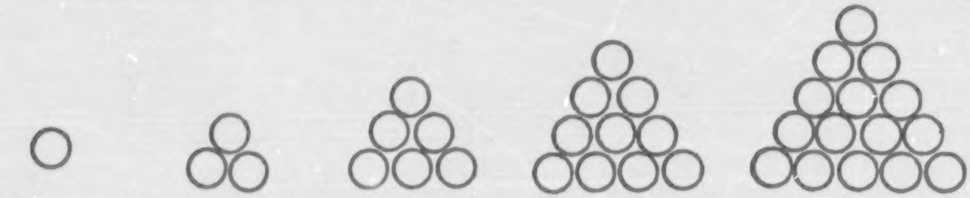
15) Consider the partial diagram for k points. How many new chords would the k+1 point add to the figure?



16) Use your result in Exercise (15) to finish PFI Part (2) for this theorem.

17) Prove by PFI that $1 + 2 + 2^2 + \ldots + 2^{n-1} = 2^n - 1$.

18) The triangular numbers are the numbers formed by placing spheres (like billiard balls) together in triangular arrays (where n is the number of balls on an edge.)



$t_1 = 1$ $t_2 = 3$ $t_3 = 6$ $t_4 = 10$ $t_5 = 15$

Derive a formula for t_n by the method of undetermined coefficients.

19) Prove your formula in Exercise (18) by PFI.

20) (from G. Polya, Induction and Analogy in Mathematics) "Look at the mathematician", said the logician. "He observes that the first 99 numbers are less than 100 and infers hence, by what he calls induction, that all numbers are less than 100."

"A physicist believes, " said the mathematician, "that 60 is divisible by all numbers. He observes that 60 is divisible by 1, 2, 3, 4, 5, and 6. He examines a few more cases, as 10, 20, and 30, taken at random as he says. Since 60 is divisible also by these, he considers the experimental evidence sufficient."

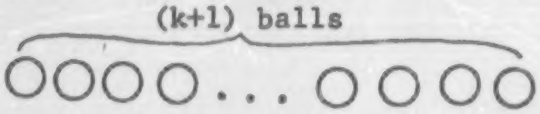
"Yes, but look at the engineers, " said the physicist. "An engineer suspected that all odd numbers are prime numbers. At any rate, 1 can be considered a prime number, he argued. Then there come 3, 5, and 7, all indubitably primes. Then there comes 9; an awkward case, it does not seem to be a prime number. Yet 11 and 13 are certainly primes. 'Coming back to 9,' he said, 'I conclude that 9 must be an experimental error.'"

What kind of induction is being applied in these examples?

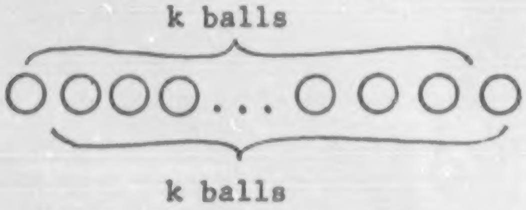
21) DIFFICULT. Here is an argument by PFI that all billiard balls are the same color. We first restate the theorem: In any set of n billiard balls, all n are the same color.

PFI Part (1) Clearly any one billiard ball is the same color as itself.

PFI Part (2) We are given that any set of k billiard balls are the same color. Here is a set of k+1 billiard balls:



We merely apply our given argument twice:



Since these sets overlap, they must all be the same color. Find the flaw in this argument. Be careful. Do not reject PFI in the process.

3.6 The Binomial Theorem

All through this course we have been interested in short-cuts, ways of saving time and energy. (In exactly this way laziness has probably always been the prime mover in man's development of civilization.) We now seek one more very useful short-cut, the means for finding the coefficients of various powers of a binomial like a + b.

You may check by multiplying (if you have nothing else to do with your time) the following products:

$$(a+b)^0 = 1$$
$$(a+b)^1 = a + b$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$
$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Those are enough to suggest some very simple patterns followed by the letters a and b. We can easily generalize them:

For $(a+b)^n$ we have

$$1\,a^n + n\,a^{n-1}\,b + \bigcirc a^{n-2}\,b^2 + \bigcirc a^{n-3}b^3 + \dots + \bigcirc a^2n^{n-2}$$

$+ n\,ab^{n-1} + b^n$ where the loops represent coefficients to be filled in. These regularities are as follows:

(1) There are n+1 terms.

(2) a starts to the n^{th} power and goes down one power in each successive term.

(3) b starts to the 0 power and increases one power in each successive term.

(4) The sum of the powers of a and b in each term is always n.

(5) Numerical coefficients increase to the middle and then decrease. The k^{th} coefficient is the same as the $(n-k)^{th}$ coefficient.

EXAMPLE 3.6-1 Express as a polynomial $(3x - 2y)^5$.

Solution: We can use what we found for $(a+b)^5$ on page 3.6-1, substituting 3x for a and (-2y) for b in each term:

$$(3x)^5 + 5(3x)^4 (-2y) + 10(3x)^3(-2y)^2 +$$
$$10(3x)^2 (-2y)^3 + 5(3x)(-2y)^4 + (-2y)^5$$

Simplifying we have:

$$243x^5 - 810x^4y + 1080x^3y^2 - 720x^2y^3 + 240xy^4 - 32y^5$$

Now let us focus on the numerical coefficients of $(a+b)^n$. To do this we will extract from the products on page 3.6 - 1 only n and the coefficients to form what has been called Pascal's Triangle:

n												
0							1					
1						1		1				
2					1		2		1			
3				1		3		3		1		
4			1		4		6		4		1	
5		1		5		10		10		5		1
6	1		6		15		20		15		6	

How can we continue this pattern? Certainly one way would be to multiply out $(a+b)^7$ and copy the coefficients, but some regularities should leap out at you. Before reading on you should try to construct row 7 by comparing it with earlier rows.

Almost certainly you will have extended the table to give the following:

5			1		5		10		10		5		1		
6		1		6		15		20		15		6		1	
7	1		7		___		___		___		___		7		1

It is clear that the first and last numbers are ones and that the next numbers in are each n. Perhaps you also discovered the pattern that produced the other coefficients. If you didn't, look back at the 15's and the 20 in row 6. Compare them with the numbers in row 5.

5			1	5	10	10	5	1	
6		1	6	15	20	15	6	1	
7	1	7	_	_	_	_	7	1	

Now almost certainly you should be able to fill in those inner spaces. They are each the sum of the numbers on either side in the row above.

EXAMPLE 3.6-2   Express as a polynomial $(2x + \frac{1}{2})^8$

Partial Solution: Continue Pascal's Triangle to find the coefficients of $(a + b)^n$

6			1	6	15	20	15	6	1
7		1	7	21	35	35	21	7	1
8	1	8	28	56	70	56	28	8	1

Thus we know $(a + b)^8 = a^8 + 8a^7b + 28a^6b^2 + \ldots$

Substituting 2x for a and $\frac{1}{2}$ for b we have

$$(2x)^8 + 8(2x)^7(\tfrac{1}{2}) + 28(2x)^6(\tfrac{1}{2})^2 + \ldots, \text{ and}$$

$$256x^8 + 512x^7 + 448x^6 + \ldots .$$

In the exercises you will have an opportunity to practice this expansion technique. We will then return to it in the next chapter.

Exercise Set 3.6

1) Construct Pascal's Triangle to $n = 10$ from $n = 0$ without reference to page 3.6 - 3.

(2 - 5) Use your answer in exercise (1) to expand and simplify the following:

2) $(a + b)^4$ 3) $(a + b)^{10}$
4) $(2x + 3y)^6$ 5) $(x - y)^5$

6) Complete Example 3.6-2 of the text.

(7 - 14) Give only the requested term for each expansion. (Refer to your answer in exercise (1).)

7) $(a + b)^9$ 4th term 8) $(a + b)^7$, 6th term
9) $(a + b)^{100}$ first term 10) $(a + b)^{500}$, second term
11) $(2x + 3y)^{10}$ second term 12) $(3x - \frac{1}{3}y)^7$, 4th term
13) $(x - y)^{30}$, second term 14) $(x - y)^{30}$, 30th term

15) Find $(205)^3$ by calculator.

16) Find $(205)^3$ by using the binomial expansion with $a = 200$ and $b = 5$.

17) Give the first three terms of $(1.04)^6$ by using the binomial expansion with $a = 1$ and $b = .04$.

18) Notice how in exercise (17) as powers of b increase the value of terms becomes increasingly small. For this reason, when b is small compared to a we have

$$(a + b)^n = a^n + na^{n-1}b + \underbrace{b^2 \cdot (\text{other terms})}_{\text{SMALL CONTRIBUTION TO TOTAL}}$$

and we have the approximation

$$(a + b)^n \approx a^n + n\, a^{n-1}\, b.$$

19) Use the method of exercise (18) to find $(20.03)^4$ to the nearest ten.

20) Complete the expansion of $(20.03)^4$ to determine your error in exercise (19).

Chapter 3 TEST

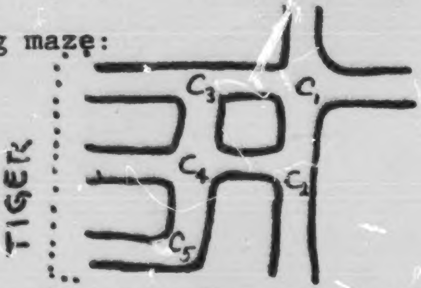
1) By iteration techniques, solve for y:

$$y = (\tfrac{1}{2})^{y^2} \qquad \text{Refer example 3.2-1}$$

Program your calculator and list the values in the table.

n	x_n	y_n
1		
2		
3		
4		
5		
10		
15		
20		
25		

2) Refer to Section 3.3; problem #9 - The Lost Lady and the Tiger. Use the following maze:



Represent the probability of escape from corner C_1 as p_1 etc. List the five recursion equations for chance of escape from each corner. Use your equations (two decimal places) to find each probability.

3) A small country has an army corps of 200 generals, 400 colonels and 800 majors. The government feels there are too many officers in the army and therefore establishes a new promotional system whereby 20% of the generals are retired each year and only 5% of the colonels are promoted. In turn 25% of the colonels are retired each year and 20% of the majors are promoted. Two hundred majors are commissioned each year but 40% of the majors are also retired each year. Eventually the total number of officers will stabilize. Use the following algorithm and complete the table. (Fix your calculator to 0 decimal places.)

Algorithm

```
1)  Remember M, C, G
2)      $N = 0$
3)      $M = 200 + .6M$
4)      $C = .20M + .75C$
5)      $G = .05C + .8G$
6)      $N = N + 1$
7)  If $N = 41$  STOP
8)  GO TO STEP 3
```

N	M	C	G
0	800	400	200
1			
2			
3			
4			
5			
10			
20			
29			
30			

4) Find the equation of the polynomial function given the following set of values:

n	f(n)
0	4
1	4
2	14
3	40
4	88
5	164

5) Prove by the principle of finite induction that:

$$1 + 4 + 7 + \ldots + (3n-2) = \frac{n(3n-1)}{2}$$

6) Expand and simplify $(3x^2 - \frac{1}{2}y)^4$

7) Find only the sixth term of the expansion: $(2x-y^2)^8$.

CHAPTER 4. APPLICATIONS OF THE FUNDAMENTAL COUNTING PRINCIPLE

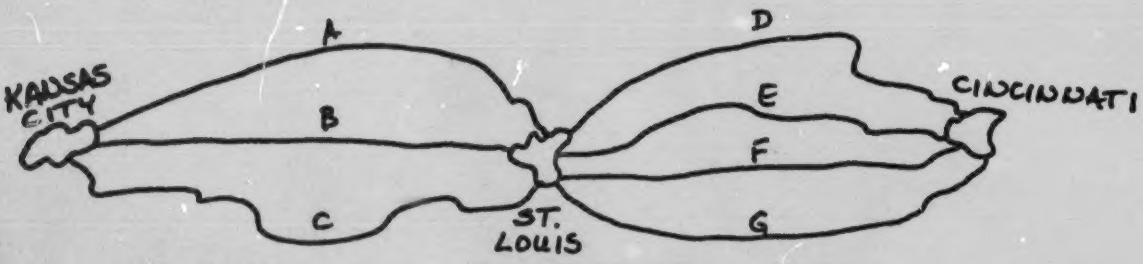
In this chapter we return to one of the most basic of mathematical skills, counting. Most of you learned to count before you even entered school; now you will have a chance to extend that skill to some sophisticated applications.

4.1 The Fundamental Counting Principle

Counting solves many many problems in the real world, but when the numbers with which we are dealing become large counting can take time. That was our original motivation for learning arithmetic operations. For example, when we have counted 15 apples in one package and 13 in another, we save ourselves the time of recounting all the apples when we find the total, 28, by addition.

Here we introduce a similar counting short cut. We will do this by means of a simple example:

EXAMPLE 4.1-1 The map below shows routes between Kansas City, St. Louis, and Cincinnati. How many different routings are there from Kansas City to Cincinnati via St. Louis?



Solution (1): By direct count we have the following routes:

1. A - D 2. A - E 3. A - F 4. A - G
5. B - D 6. B - E 7. B - F 8. B - G
9. C - D 10. C - E 11. C - F 12. C - G

There are a total of 12 routings.

Solution (2): By analysis:

For Route A there are 4 connecting routes (D,E,F,G)
For Route B there are 4 connecting routes (D,E,F,G)
For Route C there are 4 connecting routes (D,E,F,G)
Thus there are 4 + 4 + 4 = 12 routings

Solution (3): By more refined analysis

For each of the 3 routes from Kansas City to St. Louis (A,B,C) there are 4 routes from St. Louis to Cincinnati (D,E,F,G).

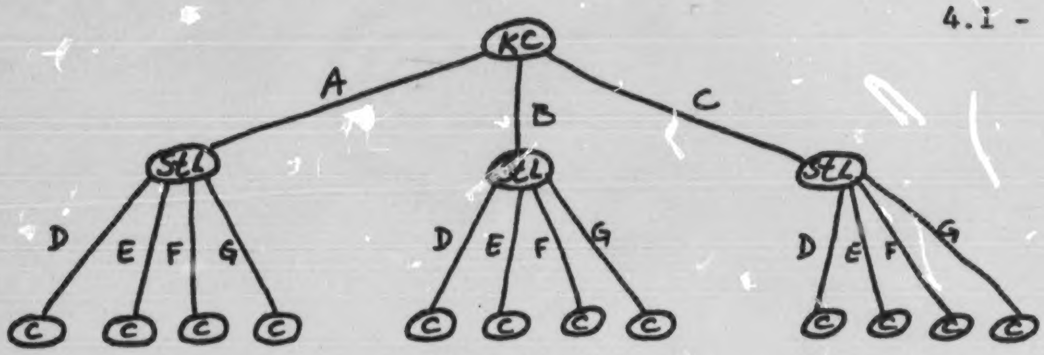
Thus there are $3 \cdot 4 = 12$ routings

- -

We will look at this same problem by means of a very useful counting device called a tree diagram:*

---

*It might better be called a root diagram because the tree and branches are almost always drawn upside down.



The tree diagram is useful in many situations which call for organization of information especially where that information takes the form of finite ordered choices. We will use this device again later.

What this example and these approaches to it are meant to justify is the following:

FUNDAMENTAL COUNTING PRINCIPLE (FCP)

If in n successive choices, the first can be done in c_1 ways, the second in c_2 ways, etc., then the total number of choices in order is $c_1 \cdot c_2 \cdot c_3 \cdot \ldots \cdot c_n$.

We have seen FCP used in the case of the routing from Kansas City to Cincinnati via St. Louis. In that example there were 3 choices of routes from Kansas City to St. Louis, 4 of routes from St. Louis to Cincinnati leading to $3 \cdot 4 = 12$ total route choices for the full trip.

Now let us see how this principle may be used in problems.

EXAMPLE 4.1-2 Most license plates in New York State have 3 digits followed by three letters. A typical plate is

258 EVR

If no restrictions are placed on these listings,[*] how many plates of this type are possible?

Solution: A useful device for FCP is a series of boxes into which the number of choices of each type is placed.

Here are the number of choices is:

10	10	10	26	26	26

By the FCP the total choices is then the product of these choices: 17,576,000 (Since New York State has about 7,000,000 registered cars, this number is sufficient - for now.) ,

---

[*] There are many such listings in practice. Many letter groups, for example, are not allowed because they bruise sensitivities.

EXAMPLE 4.1.3 How many different 5 letter words may be spelled with the letters A,B,C,D,E, and F, if no letter is repeated in any word? (Note: In mathematics a word need have no dictionary meaning. Here, for example, ABCDE is a word.)

Solution: Using the choice boxes:

6	5	4	3	2

Notice that the first letter chosen is any one of six, but the second cannot duplicate the first so only five are left.

By FCP, we have a total of $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$ words.

EXAMPLE 4.1.4 How many of the words in Example 4.1-3 begin with A and end with B?

Solution: Fill in the restricted boxes first:

A				B
1				1

Now fill in the other boxes.

1	4	3	2	1

By FCP we have a total of $1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 24$ such words.

EXAMPLE 4.1-5 How many three or four letter words may be found with the letters A,B,C,D,E, and F? No repeated letters allowed.

Solution: Consider the three letter words and the four letter words separately.

6	5	4

6	5	4	3

There are 120 three letter words and 360 four-letter words or a total of 480 words in all.

Notice in the last example that we added 120 and 360. FCP was not the technique to be used here. You must be very alert to note this difference. Often the word "or" suggests when addition is in order.

Exercise Set 4.1

1) List the words that can be formed under the conditions of Example 4.1-4. Are there 24?

2) Under the conditions of Example 4.1-1, how many different round trips, Kansas City to Cincinnati, are possible?

3) Under the conditions of Example 4.1-1, how many different round trips are possible, Kansas City to Cincinnati, if no route is driven twice on a trip?

4) Construction is being done on several of the routes in Example 4.1-1. No eastbound traffic is allowed on routes B and F, and no westbound traffic is allowed on route D. How many round trips without repeating a route are possible?

4.2 Probability

If you happen to be a sports enthusiast, you most likely are familiar with the idea of odds, relating to teams or individuals participating in athletic contests.

For instance, you may read that a football team has 7-1 odds of going to the Super Bowl, or a horse has 5-2 odds of winning a race.

What does this mean exactly?

Odds are against the team going to the Super Bowl or against the horse winning the race. Thus if the football season was run through 8 times, the team with 7-1 odds would make it once and fail 7 times. In the race, the horse at 5-2 would win twice and lose 5 times.

If we talk of probability however, the reference is for an event occurring.

> Probability is the ratio of the event happening divided by the total number of events.

The football team's probability of attending the Super Bowl is $\frac{1}{1+7}$ or $\frac{1}{8}$ while the probability of the horse winning is $\frac{2}{2+5}$ or $\frac{2}{7}$.

Basically there are two types of probability:

1) Predicting an occurrence based on past studies or experiences such as forecasting the weather or life expectancy.
2) Predicting events established by known facts. This involves experiments such as the roll of a die, draw of a card, flip of a coin or spin of a wheel. In the exercises we will see ways of shortcutting the physical work.

The number of outcomes "for" an event must be less than or equal to the total possible outcomes. Clearly, it is not possible to have a probability greater than one. Also the least probability possible is zero.

EXAMPLE 4.2-1 From life insurance records, given 100,000 20 year olds, 47,700 will be living at the age of 70. What is the probability of a 20 year old living to the age of 70?

Solution: The number "for" is 47,700

The total is 100,000

The probability is $\frac{47,700}{100,000}$ of .4770

EXAMPLE 4.2-2 If the letters A,H,M,T are arranged in any order, what is the probability that this word will spell MATH?

Solution: By The Fundamental Counting Principle there are $4 \cdot 3 \cdot 2 \cdot 1$ or 24 different arrangements only one of which spells MATH. The probability of spelling this word would be $\frac{1}{24}$ or .0417.

EXAMPLE 4.2-3

a) Given the letters E, S, T, T how many recognizably different ways can these letters be arranged?

b) What is the probability of arranging these letters in such an order as to spell the word TEST?

Solution: a) By the FCP we have 24 arrangements, however if the two T's are switched there is no recognizable different spelling. Therefore we have 24 ÷ 2 or 12 different spellings.

b) The probability of spelling TEST would be 1/12 or .0833.

EXAMPLE 4.2-4 When playing Backgammon what is the probability of rolling "double sixes" with the dice?

Solution: By the Fundamental Counting Principle there can be $6 \cdot 6$ or 36 outcomes when rolling a pair of dice. Of these 36 different results, only one is a double six. Therefore the probability of rolling a double six is $\frac{1}{36}$ or .0278.

Alternate Solution: The probability of rolling a six on the first die is $\frac{1}{6}$. The probability of rolling a six on the second is also $\frac{1}{6}$. The probability of a double six is $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$. In general, if the probability of an event is P_1 and after this has occurred the probability of a second event is P_2, then the probability of both happening is $P_1 \cdot P_2$.

In many probability experiments a "random digit table" may be of value. If 50,000 digits were listed at random, how many of these digits would you expect to be 9's? How many would you expect to be 2's? etc. Of course you most likely would expect to have 5,000 of each digit, but this would be quite rare. Actually if 6 of the digits were within 67 of the 5,000, the table would be good enough to use.

An algorithm for a crude "random digit generator" follows. It does not have theoretical support.

To generate 5 digits at a time, FIX 5.

```
1.  Remember N
2.  R ← (ln N)^2    (ln will be explored in Chapter 6)
3.  Remember R
4.  Take the fractional part of R and display
5.  N ← R, Go to Step 2
```

Select a "seed" number N, $N > 0$, $N \neq 1$ in order to generate the random digits.

For the seed number $N = 1.23$ the following 200 digits are generated. This process could be continued to generate as many digits as desired.

04285	92210	26594	75979	03054
00090	11026	16376	39246	00184
92309	42763	72165	10642	01931
60271	91500	00789	44419	95164
27960	76840	03687	00131	04952
32861	08762	83513	80683	34999
10216	00946	72077	47574	05681
62678	93269	00486	38441	19465

Suppose we desire to sample the grade point averages of 100 seniors in a graduating class of 500. We may assign each student a three digit number 001, 002, ... 500. Using the random digit table we may take the first three digits of each 5 digit number in Column 3. We would obtain the numbers 265, 163, 721, 007, etc. Students 265, 163 and 007 would be checked. The three digit number 721 would be ignored. This process, using three digit combinations in the various columns, would be continued until 100 students were selected.

The Birthday Problems:

What is the probability that at least two people in your mathematics classroom will have the same birthday? (Year of birth need not match.)

To establish an algorithm, consider the probability that a six will not show up on a single throw of a die.

The probability a six will show up is $\frac{1}{6}$. Therefore the probability that the six will not show up is $1-\frac{1}{6}=\frac{5}{6}$.

The probability of you and your closest friend in the mathematics class having the same birthday (non-leap year) is $\frac{1}{365}$ or .0027. The probability that the two of you do <u>not</u> have the same birthday is $1-\frac{1}{365}=\frac{364}{365}$.

Let's add a third person. If you and your friend have <u>two</u> different birthdates, the probability of the third person matching one of these two dates is $\frac{2}{365}$. The probability of the third person <u>not</u> matching one of these dates is $1-\frac{2}{365}=\frac{363}{365}$.

Referring to the alternate solution for example 4.2-4, the probability of all three people <u>not</u> having the same birthdate can be written:

$$\frac{364}{365}\cdot\frac{363}{365}= 9918 \qquad n = 3$$

Therefore the probability for the same birthdate is 1-.9918 = .0082.

For $n = 4$ we have $1-\frac{364}{365}\cdot\frac{363}{365}\cdot\frac{362}{365}=\underline{.0164}$

An algorithm to generate the probability of two people having the same birthdate based on the number of people present follows.

N = number of people present

```
1) Let A = 1, N = 2
2) X ← $\frac{366-N}{365}$
3) A ← A · X
4) P ← 1 - A, Display N, P
5) N ← N + 1
6) Go to Step 2.
```

Using the algorithm, complete the following table of values:

n	2	3	4	5	6	7	8	9	10
p	.0027	.0082	.0164						

n	20	21	22	23	24	25	30	40	50	60
p										

Exercise Set 4.2

1) Assuming the letters B A D in that order, are allowed on New York State license plates, what is the probability of receiving a plate with these letters in that particular order? See Section 4.1.

2) Given the letters A, C, H, I, M, R, S, S, T, what is the probability of arranging all of these letters to spell the word Christmas?

3) Given the letters A, A, H, H, K, K, N, U, what is the probability of arranging these letters to spell the word Hanukkah?

4) One your way to school each day there are three signal lights. Each light stays green in your favor for 30 seconds and red for 30 seconds. What is the probability you will not be stopped by a red light going to school?

5) In exercise 4), change the time for the green light in your favor to one minute. What is the probability you will not be stopped?

6) a) A card is drawn at random from an ordinary bridge deck of 52 cards. Find the probability that this card is an "honor" card, that is an Ace, King, Queen, Jack or Ten.

b) Find the probability that the card drawn is not an "honor" card, that is a 2, 3, 4, 5, 6, 7, 8, or 9?

c) From a) and b) above, list a conclusion regarding the probability of an event occurring, compared to that same event not occurring.

7) Using the random digit algorithm and your own selection for N, generate a random digit table of 200 digits.

8) Count the number of times each digit is listed in your random digit table.

9) Combine the totals from exercise 8) for all members of the class.

10) Using your random digit table:

a) Find an alternative to flipping a coin 100 times.

b) Explain a method to use for a replacement to actually rolling a pair of dice.

11) a) For 10a), how many of the hypothetical flips were heads?

b) For 10b), how many of the first 36 rolls are doubles?

12) Use your random digit table or the algorithm to select three names at random from a telephone book.

13) Using a random digit generator, suggest a method for drafting individuals to the military service.

14) How many people would have to be present in a room in order that the probability of two of them having the same birthdate would be greater than one-half?

4.3 Permutations

As a senior, you most likely are as conscientious a student as you were two or three years ago.

Suppose you plan to work on four different homework assignments this evening but cannot decide on which order to do this work.

You may want to start with Math, followed by English, then Social Studies and finally Science. Then again it may be English first, then the Math, a little Science and finally Social Studies.

How many different arrangmenets are possible?

By the F. C. P. you will have $4 \cdot 3 \cdot 2 \cdot 1$ or 24 different possibilities.

When arranging items in a line or a definite order as we did above, we use the word permutation to describe that precise arrangement.

> A permutation of a number of objects is any arrangement of the objects into a distinguishable order.

20 people may be arranged in a line $20 \cdot 19 \cdot 18 \cdot 17 \ldots 3 \cdot 2 \cdot 1$ ways.

Nine baseball players may be arranged in $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ different batting orders.

n objects can be arranged in $n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1$ ways.

Because of the long sequence of factors, we use a special symbol for arrangements such as the above.

[*] The product of all the whole numbers from 1 to n is called n factorial denoted by n!

$$n! = 1\cdot 2\cdot 3 \ldots (n-2)(n-1)(n)$$

OR

$$n! = n(n-1)(n-2) \ldots 3\cdot 2\cdot 1$$

OR

$$n! = n(n-1)!$$

The 20 people may be arranged 20! ways. The 9 baseball players may be arranged 9! ways.

An Algorithm may be written to generate the growing factorial function.

```
1)  Let $N = F = 1$
2)  Display N
3)  $F \leftarrow F \cdot N$,  Display F
4)  $N \leftarrow N + 1$
5)  Go to 2.
```

---

[*] A special case is 0! which is defined to equal 1.

Using the algorithm, complete the following table:

n	n!
1	1
2	2
3	6
4	
5	
6	
7	
8	

n	n!
8	
9	
10	
11	
12	
13	
14	
15	

EXAMPLE 4.3-1 How many different ways may all the cards from a bridge deck be dealt out?

Solution: Since there are 52 cards in a bridge deck, the cards may all be dealt out in 52! different ways.

To compute the value of n! for large values of n, an algorithm may be written.

```
1) Remember N
2) Let M = 1
3) T ← M·N
4) N ← N-1
5) If N > 0  Go to Step 8
6) Display T
7) STOP
8) T ← T·N
9) Go to Step 4
```

Writing a program from the algorithm when N = 52, yields $8.065\cdot10^{67}$.

Now we will develop the rule for the number of permutations of n different things taken r at a time denoted by $P(n,r)$ $n \geq r$.

EXAMPLE: How many different three letter words may be formed from the work: POKER?

Solution: We may select any of the 5 letters for the first element of our word, the second element in 4 ways and the third in 3 ways. This yields $5 \cdot 4 \cdot 3$ or 60 different 3 letter words.

This is a permutation of 5 things taken three at a time or $P(5,3) \cdot P(5,3) = 60$.

EXAMPLE: Find $P(n,r)$

Solution: The first element may be filled in n ways. This may be written $n-0$, the second element $(n-1)$ ways, continuing until we have written r elements or factors.

$$\begin{array}{lll} \text{1st factor} & & (n-0) \text{ or } n \\ \text{2nd} & " & (n-1) \\ \text{3rd} & " & (n-2) \\ \vdots & & \vdots \\ r^{th} \text{ factor} & & (n-(r-1)) \text{ or } (n-r+1) \end{array}$$

By the Fundamental Counting Principal:

$$\boxed{P(n,r) = n(n-1)(n-2)\ldots(n-r+1)}$$

Permutation Algorithm

```
1)  Remember N, R, P = 1
2)  $M \leftarrow N-R+1$, Remember M
3)  If $M \leq N$ GO TO STEP 6
4)  DISPLAY P
5)  STOP
6)  $P = M \cdot P$
7)  $M = M + 1$
8)  GO TO STEP 3
```

Factorial notation may be used for permutations.

Let us multiply the right side of the permutation equation by 1 in disguise:

$$P(n,r) = n(n-1)(n-2)\ \dots\ (n-r+1) \cdot \frac{(n-r)!}{(n-r)!}$$

$$(n-r)! = (n-r)(n-r-1)(n-r-2)\ \dots\ 3 \cdot 2 \cdot 1$$

Therefore $P(n,r) = \dfrac{n(n-1)(n-2)\ \dots\ (n-r+1)(n-r)(n-r-1)\dots 3 \cdot 2 \cdot 1}{(n-r)!}$

Which simplifies to:

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}}$$

Exercise Set 4.3

1) Using the Permutation Algorithm, write a program that may be used to find P(n,r).

2) Snow White and the Seven Dwarfs are planning to go tobogganing.
   a) How many ways can they ride an 8-person (or 1-person, 7 dwarfs) toboggan?

b) How many ways can they ride if they have a toboggan that will hold only 5 of them?

c) How many ways can they ride the 8-person toboggan if Grumpy refused to ride?

d) How many ways can they ride an 8-person toboggan if Bashful refuses to go first?

e) If Prince Charming comes along with a 9-person toboggan, how many ways can they all ride?

3) Most telephone numbers have three digits followed by four digits. Suppose no repetition of the first three digits is allowed.

a) What is the maximum number of telephone numbers?

b) What is the probability a telephone number will only have odd digits?

4) Using the first definition of $P(n,r)$

$$P(n,n) = n(n-1)(n-2)\ \ldots\ (n-(n-1))$$
$$= n(n-1)(n-2)\ \ldots\ (1) = n!$$

a) Use the second definition of $P(n,r) = \frac{n!}{(n-r)!}$ to find $P(n,n)$.

b) What does this suggest about defining $0!$?

5) Show that $\frac{1}{P(n,1)} + \frac{1}{P(n,2)} = \frac{1}{n-1}$

6) a) How many words can be formed from the letters of the word HYPERBOLA taken all together?

b) In how many of the words in a) will the letters hy in that order, occur together?

c) How many of the words in a) will the letter h and y not occur together?

7) Generally we label polygons with Capital letters at the vertices. What is the probability of labeling a triangle with the letters A, B, C in clockwise order if all letters of the alphabet are equally likely to be selected at random?

8) a) Given the digits 1, 1, 2, 2, 3, 3, 4, 4, how many different seven-digit numbers may be written?

b) How many numbers greater than 3,000,000 may be written?

c) What is the probability of writing a number greater than 3,000,000?

d) What is the probability of writing a number greater than 4,400,000?

9) Given $P(n,r) = \frac{n!}{(n-r)!}$

Find $P(n,r+1)$ in terms of $P(n,r)$

10) There may be times when it is necessary to calculate $P(n,r)$ for $r = 0, 1, 2 \ldots n$. Using the result from exercise 9) an algorithm may be written.

```
1)   Remember N , R
2)   $M \leftarrow N + 1$
3)   If $R = 0$  then $P = 1$  because $P(n,0) = 1$
4)   Display P
5)   If $P = 0$   Go to Step 10
6)   $R \leftarrow R + 1$
7)   $T \leftarrow M - R$
8)   $P \leftarrow P \cdot T$
9)   Go to Step 4
10)  STOP
```

Write a program to complete the following table for n = 9.

11)

r	P(n,r)
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

4.4 Combinations

Let's go back to doing homework assigments as mention in Section 4.3.

If a student decides to study three of four academic courses over the week-end, in how many ways can this student select the subjects without regard to order?

If the order was important, we would have P(4,3) = 24.

Since we are not concerning ourselves with the order of which subject is to be studied first, second or third, the number of ways is reduced.

If the four academic courses are Math, Science, English and Social Studies there can only be four different selections.

Math, English, Social Studies
Math, English, Science
Math, Science, Social Studies
English, Science, Social Studies

We say there are 4 combinations

Combination Definition:

> A Combination of a number of objects is a selection of these objects without regard for order.

We denote a combination of n objects taken r at a time as C(n,r), thus from above

$$C(4,3) = 4$$

Going back to the study assignment:

Let M = Math, E = English, H = Social Studies, S = Science

Combinations	Permutations
M,E,H	MEH, MHE, EMH, EHM, HEM, HME
M,E,S	MES, MSE, EMS, ESM, SEM, SME
M,S,H	MSH, MHS, SMH, SHM, HSM, HMS
E,S,H	ESH, EHS, SEH, SHE, HSE, HES

It is evident that there are 24 Permutations of the 4 subjects taken 3 at a time. These 24 Permutations may be obtained by rearranging each Combination in 3! ways.

Thus:
$$C(4,3)\cdot 3! = P(4,3)$$
$$C(4,3)\cdot(3\cdot 2\cdot 1) = (4\cdot 3\cdot 2)$$
$$C(4,3) = \frac{4\cdot 3\cdot 2}{3\cdot 2\cdot 1} = 4$$

In general, each combination of r objects can be arranged in $r!$ ways. Therefore, each $C(n,r)$ yields $C(n,r)\cdot r!$ permutations. Therefore, $C(n,r)\cdot r! = P(n,r)$

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!}\cdot\frac{1}{r!}$$

$$\boxed{C(n,r) = \frac{n!}{r!(n-r)!}}$$

EXAMPLE 4.4-1 In how many ways can a bridge hand of 13 cards be selected from a standard deck of 52 cards?

Solution: Since the order of receiving the cards does not matter, we have a combination problem.

$$C(52,13) = \frac{52!}{13!\cdot 39} = \frac{52\cdot 51\ \ldots 40\cdot 39!}{13!\cdot 39!} = 635,\ 013,\ 559,\ 600$$

EXAMPLE 4.4-2 Suppose 21 astronauts are available for the lunar landing program and 12 have had orbital experience.

a) How many crews of three can be made up?

b) What is the probability that the crew of three will be made up of at least one experienced and one inexperienced person?

Solution: a) $C(21,3) = \frac{21!}{3! \cdot 18!} = 1330$

b) There are two cases to be considered:

1. Having two experienced and one inexperienced OR

2. One experienced and two inexperienced astronauts to make up the crew of three.

For (1) we have C(12,2) or 66 ways of choosing two astronauts from the 12 with experience, and for each choice, there are C(9,1) or 9 ways of choosing one astronaut from the 9 without experience. Therefore we have $C(12,) \cdot C(9,1) = 66.9 = 594$ crews for (1).

Similarly for (2) there are $C(12,1) \cdot C(9,2) = 12 \cdot 36 = 432$ possible crews.

The probability of having at least one experienced and one experienced astronaut on the crew is:

$$\frac{594 + 432}{1330} = .7714$$

To use your calculator to find combinations, an algorithm may be developed.

Combination Algorithm: $C(n,r)\ r \neq 0$

```
1)   Remember N,R
2)   $B \leftarrow R,\ T \leftarrow N$
3)   $R \leftarrow R-1$
4)   If $R = 0$   Go to Step 9
5)   $B \leftarrow B \cdot R$
6)   $N \leftarrow N-1$
7)   $T \leftarrow T \cdot N$
8)   Go to Step 3
9)   $C = \frac{T}{B}$
10)  STOP
```

<u>Exercise Set 4.4</u>:

1) Write a program using the Combination Algorithm and check the result for EXAMPLE 4.4-1.

(2 - 7) Use the program to evaluate each of the following:

2) $C(100,3)$ 3) $C(100,97)$ 4) $C(52,5)$

5) $C(52,47)$ 6) $C(15,10)$ 7) $C(15,5)$

(8-13) Find the value of r that makes each of the following true.

8) $C(10,3) = C(10,r)$ 9) $C(8,7) = C(8,r)$ 10) $C(6,4) = C(r,2)$

11) $C(12,7) = C(r,5)$ 12) $C(r,6) = C(r,2)$ 13) $C(n,r) = C(n,x)$; r in terms of n <u>and</u> x.

(14-16) A Mathematics Class contains 14 boys and 9 girls:

14) In how many ways may a committee of three students be selected from this class?

15) How many of these committees contain only boys?

16) How many of these committees contain only girls?

17) Arrange the following in increasing order: 10!, C(7,3), P(7,3), 7! + 3!, $\frac{20!}{2!}$

(18-21) What is the probability when receiving 5 cards dealt from a deck of 52 cards that the five cards

18) Will contain four cards of the same kind?

19) Will contain three cards of one kind and two of another? ("full house")

20) Will contain five cards from the same suit? ("flush"). Note: exclude straight flushes, (5 card sequences).

21) Rank the hands of "4 of a kind", "flush" and "full house". Give a reason for your ranking.

4.5 Binomial Expansion Again

In Section 3.6 the binomial theorem was developed and the numerical coefficients for $(a+b)^n$ were obtained from Pascal's Triangle.

n	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1

Let us relate these numbers to combinations. For the row where n = 5, we will find the values of C(5,r) where r = 0,1,2,3,4,5

$$C(5,0) = C(5,5) = \frac{5!}{0!5!} = 1$$

$$C(5,1) = C(5,4) = \frac{5!}{1!4!} = 5$$

$$C(5,2) = C(5,3) = \frac{5!}{2!3!} = 10$$

These values correspond to the six values for the row where n = 5.

We may therefore replace this row with:

C(5,0)   C(5,1)   C(5,2)   C(5,3)   C(5,4)   C(5,5)

Recall that to complete the next row in Pascal's Triangle, we added previous values:

n = 5 1 5 10 10 5 1

n = 6 1 6 15 20 15 6 1

The values in the row for n = 6 also can be written in Combination notation:

C(5,0) C(5,1) C(5,2) C(5,3) C(5,4) C(5,5)

C(6,0) C(6,1) C(6,2) C(6,3) C(6,4) C(6,5) C(6,5)

$$\text{thus}\quad \begin{aligned} &C(5,0) + C(5,1) = C(6,1)\\ &C(5,1) + C(5,2) = C(6,2)\\ &C(5,2) + C(5,3) = C(6,3)\\ &\text{etc.}\end{aligned}$$

In general we may state: $C(n,r-1) + C(n,r) = C(n+1,r)$

The proof will be left for an exercise.

We may now rewrite the binomial expansion in Combination Notation:

$$(a+b)^n = C(n,0)a^{n-0}b^0 + C(n,1)a^{n-1}b^1 + C(n,2)a^{n-2}b^2 + \dots$$
$$+ C(n,r)a^{n-r}b^r + \dots + C(n,n-1)a^1b^{n-1} + C(n,n)a^{n-n}b^n$$

Take special note of the exponents of a and b. The exponent for b is the same as r; exponent for a is n-r.

Therefore to obtain a particular term in an expansion, say the r^{th} term, the coefficient is obtained by finding C(n,r-1).

EXAMPLE: Find the 30^{th} term only of $(a+b)^{52}$

Solution: The coefficient of the 50th term will be the same as the coefficient of the 2nd term as C(52,50) = C(52,2).

Either way the 50th term is $C(52,50)a^2b^{50}$ OR $1326a^2b^{50}$

If an entire expansion of a binomial is necessary, an algorithm may be written to calculate a row of Pascal's Triangle. We will use the property:

$$C(n,r+1) = C(n,r) \cdot \frac{n-r}{r+1}$$

$$C(n,r+1) = \frac{n!}{(r+1)!(n-r-1)!} = \frac{n(n-1)\ldots(n-r+1)(n-r)(n-r-1)!}{(r+1)r! \cdot (n-r-1)!} =$$

$$\left[\frac{n(n-1)\ldots(n-r+1)}{r!}\right]\frac{(n-r)}{r+1}$$

but $C(n,r) = \dfrac{n(n-1)\ldots(n-r+1)(n-r)!}{r!\,(n-r)!} = \dfrac{n(n-1)\ldots(n-r+1)}{r!}$

Substituting we have: $C(n,r+1) = C(n,r) \cdot \dfrac{n-r}{r+1}$

Pascal's Triangle Algorithm $N \leq 36$

```
1)  Remember N
2)  M ← N+1
3)  Let R = 0   then C = 1
4)  Display C
5)  If C = 0   Go to Step 11
```

```
6)  $R \leftarrow R + 1$
7)  $C \leftarrow \frac{C}{R}$
8)  $T \leftarrow M - R$
9)  $C \leftarrow C \cdot T$
10) Go to Step 4
11) STOP
```

Exercise Set 4.5

1) Using the algorithm, write a program to find a row of Pascal's Triangle. Check your results by verifying the row when n=6.

(2-7) Use your program to help expand the following:

2) $(\frac{a}{2} + 1)^8$

3) $(1 + \frac{1}{2}x^2)^8$

4) $(x^2 - x^3)^9$

5) $(a - ax)^9$

6) $(3c + 6)^7$

7) $(2 + 4m)^7$

8) Prove $C(n, r-1) + C(n,r) = C(n + 1, r)$

4.6 Summation Notation

There is an abbreviated way, called Sigma notation, to write finite sums of a recursive nature such as those sums produced by the binomial theorem.

Of course, the Greek letter Sigma $\sum$ will be used.

In Chapter 2 we wrote the series sum as $\$_n = a_1 + a_2 + a_3 + \ldots a_r$

We may abbreviate this with $\sum_{k=1}^{n} a_k$ where a_k represents some function and k is the domain.

It is read as the sum of all a_k as k goes from 1 to n.

EXAMPLE 4.6-1 Evaluate $\sum_{k=1}^{4} \frac{1}{k}$

Solution: Here $a_k = \frac{1}{k}$

$$\sum_{k=1}^{4} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

EXAMPLE 4.6-2 Evaluate $\sum_{k=4}^{7} k(k-2)$

Solution: Here $a_k = k(k-2)$ and k begins with 4.

Therefore $\sum_{k=4}^{7} k(k-2) = 4(2)+5(3)+6(4)+7(5)=8+15+24+35$

EXAMPLE 4.6-3 Write the following in Sigma notation:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$

Solution: $1 = (\frac{1}{2})^0,\ \frac{1}{2} = (\frac{1}{2})^1,\ \frac{1}{4} = (\frac{1}{2})^2$, etc.

$$(\tfrac{1}{2})^0 + (\tfrac{1}{2})^1 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + (\tfrac{1}{2})^4 =$$

$$\sum_{k=0}^{4} (\tfrac{1}{2})^k \quad \text{OR} \quad \sum_{k=1}^{5} (\tfrac{1}{2})^{k-1}$$

The binomial expansion may be written with sigma notation:

$$(a+b)^n = C(n,0)\cdot a^n b^0 + C(n,1)\cdot a^{n-1}b^1 + \ldots C(n,k)\cdot a^{n-k}\cdot b^k + \ldots C(n,n)a^{n-n}b^n$$

Therefore: $(a+b)^n = \sum_{k=0}^{n} C(n,k)\cdot a^{n-k}\cdot b^k$

Exercise Set 4.6

(1 - 8) Find the numerical values of each of the finite sums:

1) $\sum_{k=1}^{5} \frac{1}{k(k+1)}$

2) $\sum_{k=0}^{4} 3^{2k}$

3) $\sum_{k=1}^{5} k^k$

4) $\sum_{k=0}^{10} \frac{2^k}{k+1}$

5) $\sum_{k=1}^{100} 5$

6) $\sum_{k=1}^{10} \frac{3}{k}$

7) $\sum_{k=2}^{6} 3k^2-5k+1$

8) $\sum_{k=2}^{12} 2^{k-2}$

(9 - 16) Replace the question mark in each of the following with the symbol =, < , or >

9) $\sum_{k=0}^{100} k^4$?

10) $\sum_{k=0}^{100} (k+2)$? $(\sum_{k=0}^{100} k) + 2$

11) $\sum_{k=0}^{99} (k+1)^2$? $\sum_{k=1}^{100} k^2$

12) $\sum_{k=0}^{100} k^2$? $(\sum_{k=0}^{100} k)^2$

(13 - 16) Express each of the following finite sums in $\sum$ notation:

13) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$

14) $\frac{3}{5} + \frac{9}{25} + \frac{27}{125} + \frac{81}{625}$

15) $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15$

16) $\frac{1}{6} + \frac{2}{11} + \frac{3}{16} + \frac{4}{21}$

4.7 Chapter 4 Test

1) In preparing a multiple choice test, a teacher writes 15 questions with four different choices for each question. How many different ways could a student answer the test? (assuming all questions are answered) 1)__________

2) In a certain high school, there are 17 English teachers, 20 Social teachers, 16 Science teachers, 14 Mathematics teachers and 10 Physical Education teachers. If a student takes all five subjects, what is the probability the student will receive the teacher of his or her choice in each subject? Fix 9. 2)__________

3) How many different eleven letter words can be formed from the letters that make up the word "Mathematics"? 3)__________

4) Solve $\frac{12!}{6!-3!}$ 4)__________

5) A poker hand consists of 5 cards dealt from a deck of 52 cards. How many different poker hands are possible? 5)__________

6) $x \neq y$ but $C(n,x) = C(n,y)$ Find a value for x in terms of n and y that makes the statement true. 6)__________

7) $x \cdot P(n,r) = C(n,r)$ Find the value of x in terms of r. 7)__________

8) Simplify completely: $\frac{1}{P(n,1)} + \frac{1}{P(n,2)}$ 8)__________

9) Find the 11^{th} term only in the expansion of $(5y + 2b)^{12}$. 9)__________

10) If $P(n,4) = 2 \cdot P(n,2)$ find n. 10)__________

11) Solve for n: $\frac{P(n,4)}{C(n-1,3)} = 60$ 11)__________

12) Evaluate:

$$\sum_{k=0}^{5} \frac{k^2}{k!}$$

12)__________

13) Write the binomial expansion $(a+b)^n$ in sigma notation. 13)__________

14) What is the probability that if three persons are in a room, none of the three have the same birth-month? 14)__________

CHAPTER 5. POLYNOMIAL FUNCTIONS

Earlier in your study of mathematics you were concerned with quadratic functions, functions of the form $ax^2 + bx + c$, and linear functions of the form $mx + b$. In this chapter you will meet more general and higher power functions. Your calculating tools will help you to deal with them.

5.1 Graphing Polynomial Functions

Given a whole number n and a variable x, then a function $P(x)$ of the type

$$\boxed{P(x) = a_n x^n + \ldots \quad a_3 x^3 + a_2 x^2 + a_1 x + a_0}$$

where $a_n \ldots a_3, a_2, a_1, a_0$ are constants and $a_n \neq 0$ will be referred to as a polynomial function of degree n[*]. If the terms are arranged in descending order of exponents, the function is said to be expressed in standard form. You may, if you wish, use y in place of $P(x)$ in representing a polynomial function.

All polynomial functions when plotted will be smooth curves without breaks. These functions may be drawn without lifting the pencil, that is, they are continuous. The domain of a polynomial function is the set of real numbers.

---

[*] If $n = 0$, zero degree or a constant function
$n = 1$, first degree or a linear function
$n = 2$, second degree or a quadratic function
$n = 3$, third degree or a cubic function etc.

As mentioned earlier, in previous courses you became quite familiar with linear and quadratic functions, therefore in this section we will focus our attention on functions of degree greater than two.

A programmable calculator is a great aid in graphing these polynomial functions. Simply write a program consisting of the proper keystrokes needed to evaluate the function, followed by $\boxed{\text{R/S}}$ or $\boxed{\text{PAUSE}}$ and a key that returns the program to the beginning. By repeatedly entering values of x, each time running the program to obtain the $P(x)$ or y value, points may be obtained for plotting the function.

If you are working with a computer or wide-paper printing programmable calculator, you can "automate" the entire process by having the machine actually draw the graph, in a somewhat crude form, using a symbol such as "*" to represent each point. It is easiest to do this sideways so that the x-axis is vertical.

Let's first consider a less demanding task that even non-printing programmable calculators can perform; choosing regularly spaced x-values within an interval say from $x = s$ to $x = t$. $[s, t]$ This is an arithmetic sequence of values with a constant difference of d. $s, s+d, s + 2d, \ldots$ We can stop the process as soon as the value exceeds t.

The algorithm for this procedure is:

```
1) Remember $s, t, d$
2) $x \longleftarrow s$, display $x$
```

* 3) Calculate P(x) and display P(x)

4) $x \leftarrow x + d$

5) If $x + d > t$ stop

6) Go to step 3

EXAMPLE 5.1.1 Sketch the cubic

$$P(x) = x^3 + \frac{3}{2}x^2 - 6x - 2$$

for the interval $[-4, 3]$ for $d = .5$

When programming a calculator, it saves calculating steps to rewrite the polynomial by factoring x from part of the expression as many times as possible. A binomial will finally be nested in a series of products. The polynomial is then said to be in nested form

$$P(x) = x^3 + \frac{3}{2}x^2 - 6x - 2$$

$$P(x) = (x^2 + \frac{3}{2}x - 6)x - 2$$

$$P(x) = [(x + \frac{3}{2})x - 6]x - 2$$

Now it is not necessary for us to square and cube x. We can instead calculate "from the inside out".

HP 33E · $R_1 \leftarrow s$, $R_2 \leftarrow t$, $R_3 \leftarrow d$

```
1    RCL  1        08   GTO  10          14   .         21   RCL  1
2    GSB  12       09   GTO  01          15   5         22   x
03   RCL  3        10   CLX              16   +         23   2
4    STO + 1       11   R/S              17   RCL  1    24   -
5    RCL  2        12   f pause or       18   x         25   f  pause  or R/S P(x
06   RCL  1              R/S (x)         19   6         26   g  RTN
7    f x > y       ------------------
                   13   1  subroutine    20   -
                           begins
```

---

* Step 3 may be handled as a subroutine contained at the end of the program.

TI 58 $R_1 \leftarrow s$, $R_2 \leftarrow t$, $R_3 \leftarrow d$

```
00   RCL          15   C                 30   RCL
01   01           16   RST               31   01
02   SBR          17   2nd Lbl           32   -
03   A            18   C                 33   6
04   RCL          19   CLR               34   =
05   03           20   R/S               35   X
06   Sum          21   2nd Lbl           36   RCL
07    01          22   A                 37   01
08    RCL         23   2nd Pause         38   -
09    01          24   +    sub-         39   2
10   x >< t       25   1    routine      40   =
11   RCL          26   .     begins      41   2nd Pause
12   02           27   5                 42   INV  sub
13   INV  2nd     28   =
14   x ≥ t        29   X
```

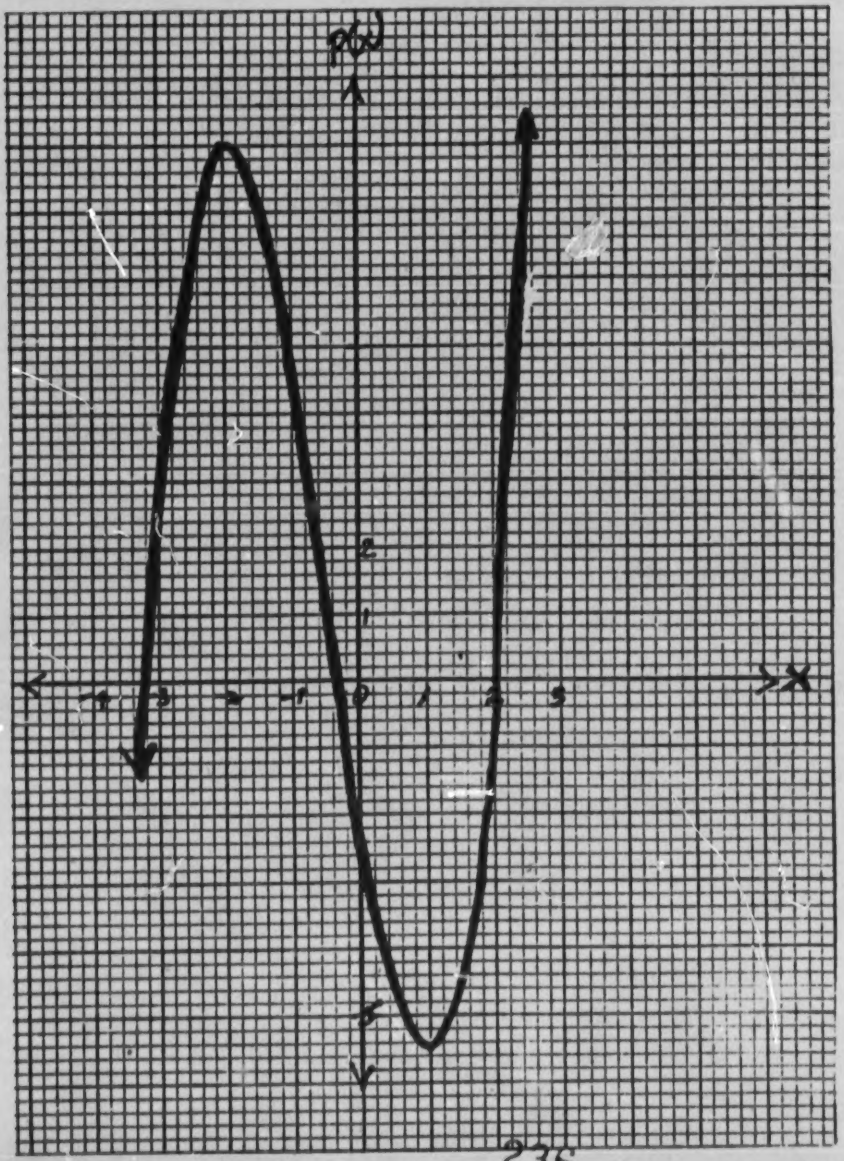
TRS 80

```
10    INPUT  S, T, D
20    X = S
30     PRINT  "X",  "P(X)"
40     Y =  X*(X*(X + 3/2)-6) -2
50     PRINT  X,  Y
60     X = X + D
70     IF  X > T    GO TO  90
80     GO TO 40
90     END
```

Running the program the following information is obtained. The graph of the function is then sketched.

x	P(x)
-4	-18
-3.5	-5.5
-3	2.5
-2.5	6.75
-2	8
-1.5	7
-1	4.5
-.5	1.25

x	P(x)
0	-2
.5	-4.5
1	-5.5
1.5	-4.25
2	0
2.5	8
3	20.5



Graphing polynomial functions with a computer

When "drawing" a graph on a cathod-ray-tube (CRT) or on a computer print-out paper we begin by obtaining information on the number of lines available and the number of characters per line. We wish to locate our first value of x (s) on the first line and move to the right a number of character spaces corresponding to the P(x) or y value, marking this point with a "*". This process is continued on a separate line for each x until our final point (t, P(t)) is printed on the last line.

Let us assume there are N spaces between the top and bottom lines, then the distance d between any two lines would be $\frac{t-s}{N}$. The sequence of x values would read

$$s, \quad s + \frac{t-s}{N}, \quad s + \frac{2(t-s)}{N}, \ \ldots \quad \underbrace{s + \frac{N(t-s)}{N}}_{t}$$

There will be N + 1 values or lines.

The y-spacing requires more thought. We desire to utilize the full width of our display, therefore we need to know the largest and smallest values for P(x). The difference between these two values in our example 5.1-1 is 20.5 - (-18) or 38.5. If our computer can display 60 characters on a line we will allot $(\frac{1}{38.5})(60)$ characters per unit on each line.

$$C = \frac{60}{38.5}$$

The y location for the lowest value (L) of the graph will correspond to $(-18)(\frac{60}{38.5})$ or (L)(C) characters. This point "*" should print on the left margin. The greatest value corresponds to $(20.5)(\frac{60}{38.5})$ characters and this point should be printed at the right-hand margin.

Note that $20.5(\frac{60}{38.5}) - (-18)(\frac{60}{38.5}) = 60$

To locate a point exactly in the middle of a line, that is for a y value of $\frac{(20.5) + (-18)}{2} = 1.25$, we would move $(1.25)(\frac{60}{38.5}) - (-18)(\frac{60}{38.5})$ or $(19.25)(\frac{60}{38.5}) = 30$ characters to the right.

In general to locate the proper 2nd coordinate, move to the right

$Y \cdot C - L \cdot C$ characters.

Point - Plotting Algorithm

s and t represent the x-interval

H and L represent the P(x)-interval (high and low respectively)

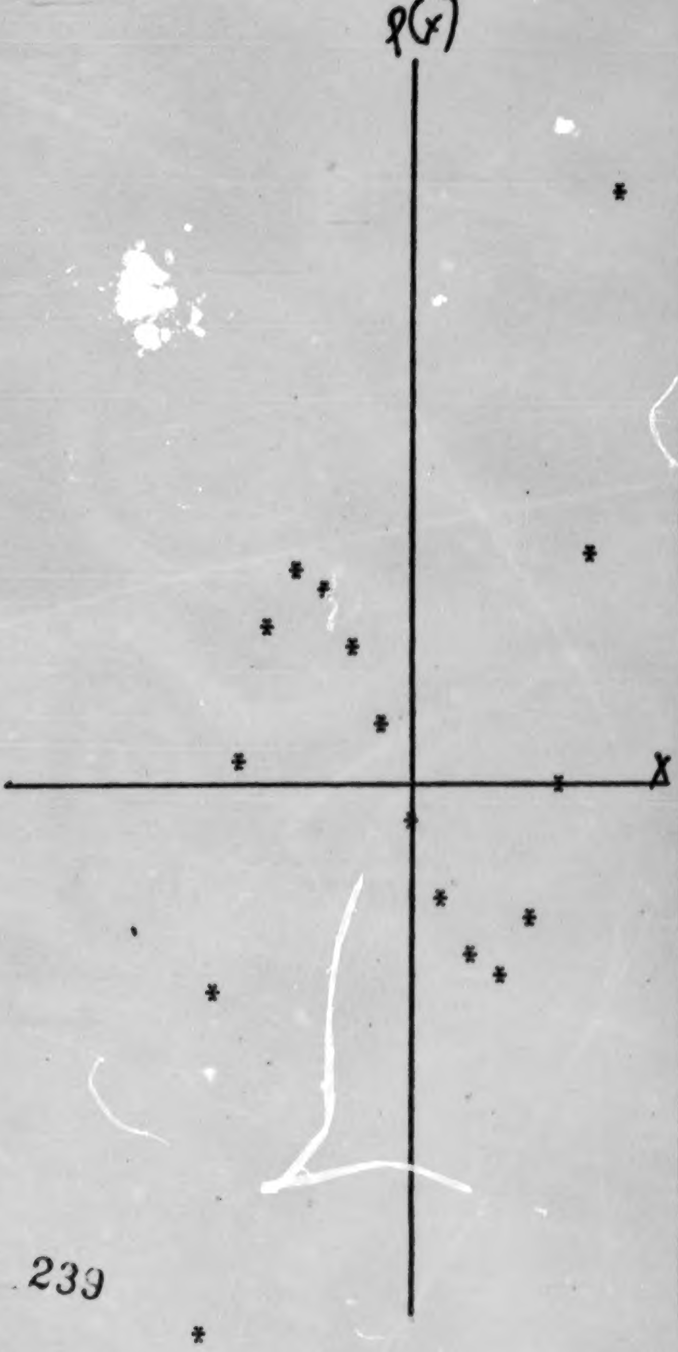
N + 1 is the number of lines available

W is the number of characters per line

```
1) Remember s, t, N, H, L, W
2) $d \leftarrow \frac{t-s}{N}$,  $x \leftarrow s$,  $C \leftarrow \frac{W}{H-L}$
3) Calculate Y by a subroutine
4) $Z \leftarrow C \cdot Y - C \cdot L$
5) Print "*" at Z units to the right (for TRS 80, Tab (Z) "*")
6) If N = 0    STOP
7) $N \leftarrow N - 1$,  $x \leftarrow x + d$    GO TO STEP 3
```

The graph of $P(x) = x^3 + \frac{3}{2}x^2 - 6x - 2$ is sketched below using a TRS 80 computer. Axis added after the run.

```
10 READ S,T,N,H,L,W
20 D=(T-S)/N
30 X=S
40 C=W/(H-L)
50 Y=X*(X*(X+3/2)-6)-2
60 Z=C*Y-C*L
70 LPRINT TAB(Z)"*"
80 IF N=0 GOTO 120
90 N=N-1
100 X=X+D
110 GOTO 50
120 DATA -4,3,14,20.5,-18,60
```



Exercise Set 5.1

1 - 7 Write a program that can be used to find the function values for each x in the interval [s, t] as listed. Use the indicated value for d. Attach the program as a subroutine and tabulate the points. Finally graph the function to the nearest tenth.

1) $A(x) = x^3 - 3x^2 - 9x + 11$ [-3, 5], d = .5

2) $B(x) = 2x^3 - 9x^2 + 12x - 5$ [-1, 4], d = .5

3) $C(x) = -x^3 + 3x^2 - 1$ [-1.5, 3.5], d = .25

4) $D(x) = x^3 - 3x^2 + 1$ [-1.5, 3.5], d = .25

5) $E(x) = 4x^4 + 2x^3 - 19x^2 - 11x + 6$

[-2.25, 2.25], d = .25

6) $F(x) = 2x^4 + 3x^3 - 7x^2 - 12x - 4$

[-2.25, 2.25], d = .25

7) $G(x) = x^5 + 2x^4 - 5x^3 - 10x^2 + 4x + 8$

[-2.2, 2.2], d = .1

5.2 Approximating Maximum - Minimum Values and Zeros of Polynomial Functions

When sketching a polynomial function we should have some idea of what interval to work with and what the graph will look like outside that interval.

By factoring out the highest power when a function is in standard form, we have, using the example from Section 5.1.1:

$$P(x) = x^3 + \frac{3}{2}x^2 - 6x - 2 \longrightarrow P(x) = x^3\left(1 + \frac{3}{2x} - \frac{6}{x^2} - \frac{2}{x^3}\right).$$

As x becomes very large (increases without bound or approaches infinity) the value in parenthesis approaches 1. This is because each fraction becomes very small as its denominator increases. For that reason $P(x)$ approaches the value of x^3 which, in this case, becomes very large. We use the following notation*

$$\text{as } x \rightarrow \infty, \quad \left(1 + \frac{3}{2x} - \frac{6}{x^2} - \frac{2}{x^3}\right) \rightarrow 1 \text{ and } P(x) \rightarrow x^3.$$

In the same manner, as $x \longrightarrow -\infty$,

$$\left(1 + \frac{3}{2x} - \frac{6}{x^2} - \frac{2}{x^3}\right) \rightarrow 1 \text{ and } P(x) \longrightarrow x^3, \text{ which is very small.}$$

More generally, given $P(x) = a_n x^n + \ldots a_3 x^3 + a_2 x^2 + ax + a_0$ by factoring out the leading term

---

*Do not confuse the notation $x \rightarrow \infty$, as x approaches infinity, with $R_1 \longleftarrow s$, which means R_1 is replaced by s.

$$P(x) = a_n x^n \left(1 + \frac{a_3}{a_n x^{n-3}} + \frac{a_2}{a_n x^{n-2}} + \frac{a_1}{a_n x^{n-1}} + \frac{a_0}{a_n x^n}\right)$$

Therefore as $|x| \longrightarrow \infty$, $P(x) \longrightarrow a_n x^n$.

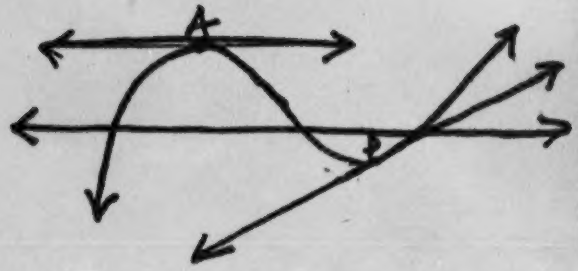
We can see that the leading term becomes the dominating value as x becomes very large or very small.

The graph of the cubic function $P(x) = x^3 + \frac{3}{2} x^2 - 6x - 2$ (as in fact every cubic function) extends indefinitely both upward and downward (the range is thus the set of real numbers). A cubic function, unlike a second degree function (parabolic graph) does not have an absolute maximum or minimum point. But the graph on page 5.1-5 suggests that the point (-2, 8) from P(x) seems to be higher than other nearby points of the graph. If this is indeed true, the point is called a relative maximum point and the function P is said to have a relative maximum $P(-2) = 8$, when $x = -2$. Similarly, P seems to have a relative minimum $P(1) = -5.5$ when $x = 1$*.

---

* At this time we cannot be sure of these relative maximum and minimum points; we can, however, test values on either side of x close to -2 or 1, comparing these P(x) values with P(-2) and P(1).

Frequently lines are drawn, intersecting a point on a graph. If the point is located on a function, only one of these lines will be tangent to the function at that particular point.



A point will represent a relative maximum or minimum if the tangent to the function at that point is parallel to the X-axis.

A zero of the function P(x) is a value r such that $P(r) = 0$.

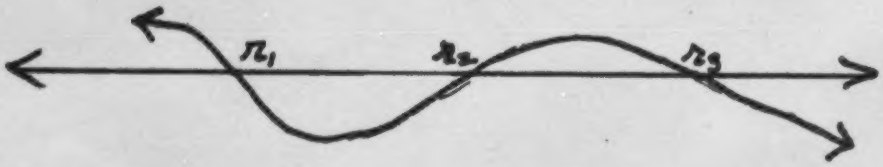
Graphically a real zero of any function is simply the value of x where the graph crosses the X-axis. If the zeros are not real, the function will not intersect the X-axis. EXAMPLE 5.1.1 shows three real values, $x = 2$ is obvious, the other two zeros may be estimated to the nearest tenth -3.2 and -.3. In Section 5.4 we will use our calculator to find these zeros to greater accuracy.

All polynomial functions may be written in factored form such as $P(x) = a_n (x - r_1)(x - r_2)(x - r_3) \ldots (x - r_n)$ where $r_1, r_2, r_3 \ldots r_n$ are members of the complex number system.

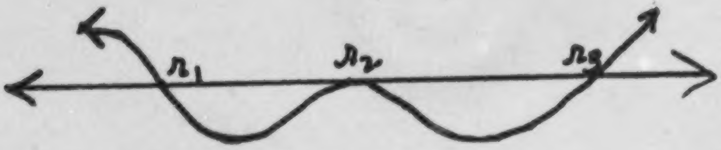
$$P(r_1) = 0, \quad P(r_2) = 0, \quad \ldots \quad P(r_n) = 0.$$

$r_1, r_2, \ldots r_n$ are the zeros of the function.

An n^{th} degree polynomial will have n factors and corresponding zeros and the equation will have n roots. Each value of r that is a real number will represent an X-intercept when P(x) is graphed.



If an even number of the real values of r are the same, the graph of the function will be tangent to the X-axis at that particular value of r.



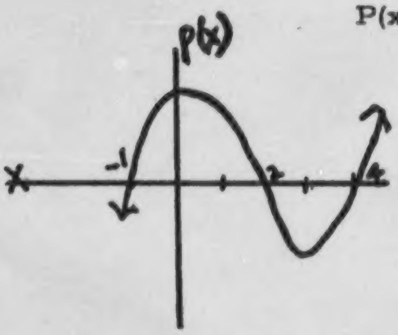
We will not take up the process of factoring polynomials at this time.

EXAMPLE 5.2.1 Find the zeros of

$P(x) = x^3 - 5x^2 + 2x + 8$

Solution: $P(x) = (x^2 - x + 2)(x - 4)$

$P(x) = (x + 1)(x - 2)(x - 4)$



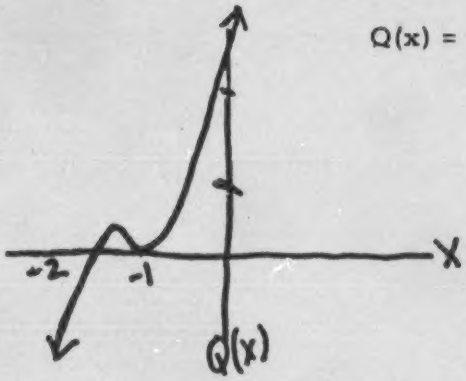
-1, 2, and 4 are the X-intercepts or zeros of the function. Relative maximum between -1 and 1. Relative minimum between 2 and 3.

EXAMPLE 5.2.2 Find the zeros of

$Q(x) = 2x^3 + 7x^2 + 8x + 3$

Solution: $Q(x) = (2x + 3)(x + 1)(x + 1)$

$Q(x) = 2(x + \frac{3}{2})(x + 1)^2$

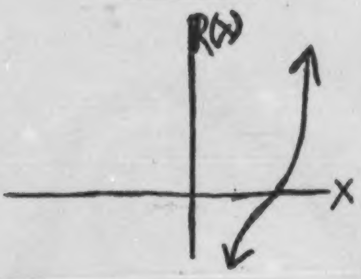


-3/2 and -1 are the zeros. -1 is said to be a zero of the function of multiplicity two. Relative maximum between -3/2 and -1.

EXAMPLE 5.2.3 Find the zeros of

$R(x) = x^3 - 3x^2 + 3x - 1 = (x - 1)^3$

Solution: $R(x) = (x - 1)^3$



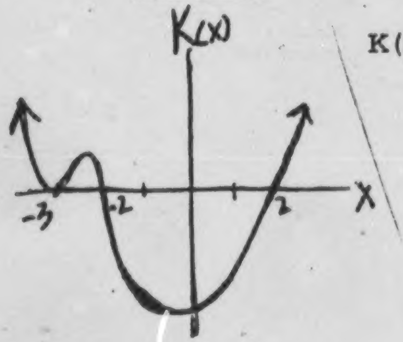
1 is a zero of multiplicity three.

EXAMPLE 5.2.4 Find the zeros of

$K(x) = x^4 - 6x^3 + 5x^2 - 24x - 36$

Solution: $K(x) = (x^2 - 4)(x^2 + 6x + 9)$

$K(x) = (x - 2)(x + 2)(x + 3)^2$



Three X - intercepts 2, -2, -3.

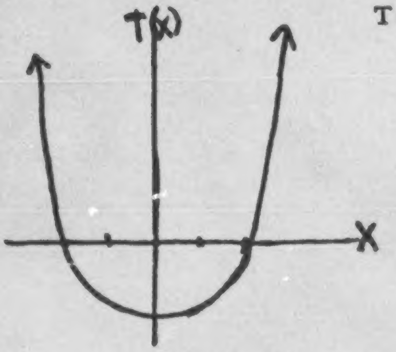
Relative maximum between -2 and -3.

EXAMPLE 5.2.5 Find the zeros of

$T(x) = x^4 + 5x^2 - 36$

Solution: $T(x) = (x^2 - 4)(x^2 + 9)$

$T(x) = (x + 2)(x - 2)(x + 3i)(x - 3i)$



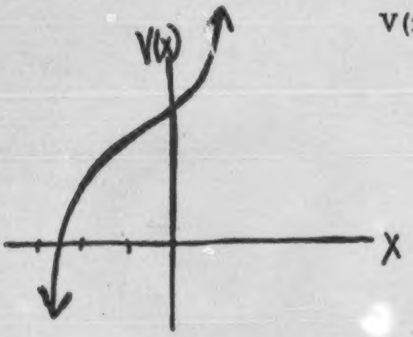
Two X- intercepts -2, 2. Note that this is NOT a parabola, T(x) is much "flatter". Note too that the complex zeros do not appear on this real graph. The zeros of this function are 2, -2, 3i, and 3i.

EXAMPLE 5.2.6 Find the zeros of

$V(x) = 2x^3 + 5x^2 + 8x + 20$

Solution: $V(x) = (2x + 5)(x^2 + 4)$

$V(x) = 2(x + \frac{5}{2})(x + 2i)(x - 2i)$



-5/2 and 2i and -2i are zeros and there is only one X-intercept.

In EXAMPLES 5.2.5 and 5.2.6 two of the values of r are complex conjugates. Remember the coefficients of a polynomial function $a_n \dots a_3, a_2, a_1, a_0$ are real numbers. Clearly complex values of r must occur in even multiples, in fact they will be conjugates (a + bi, a - bi).

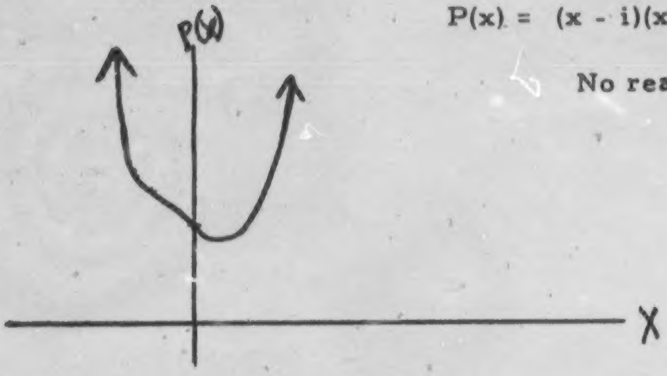
EXAMPLE 5.2.7 Find the zeros of

$P(x) = x^4 - 4x^3 + 6x^2 - 4x + 5$

Solution: $P(x) = (x^2 + 1)(x^2 - 4x + 5)$

$P(x) = (x - i)(x + i)(x - (2 + i))(x - (2 - i))$

No real zeros, no X-intercepts.



EXAMPLE 5.2.8 Find the zeros of

$W(x) = -x^3 + 3x^2 - 4$

Solution: $W(x) = -(x^3 - 3x^2 + 4)$

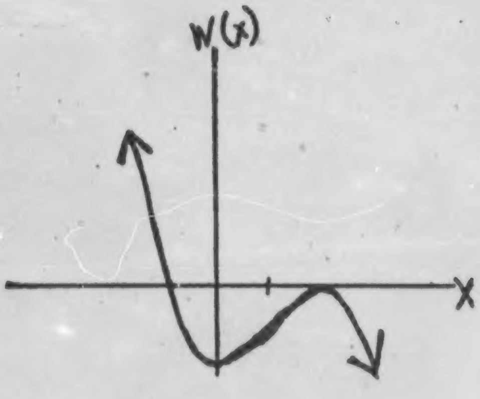
$W(x) = -(x + 1)(x^2 - 4x + 4)$

$W(x) = -(x + 1)(x - 2)(x - 2)$

as $x \longrightarrow \infty$ $W(x) \longrightarrow -\infty$

as $x \longrightarrow -\infty$ $W(x) \longrightarrow \infty$

-1, 2 are the intercepts. Relative maximum at 2. Relative minimum at 0.



Exercise Set 5.2

1) For Q(x) in EXAMPLE 5.2.2

   a) Estimate the relative minimum point

   b) Estimate the relative maximum value for x

2) For W(x) in EXAMPLE 5.2.8

   a) Estimate the relative minimum point

   b) Estimate the relative maximum point

3) For A(x) in Exercise Set 5.1 - 1

   a) Estimate the relative maximum and relative minimum point

   b) Estimate the three zeros to the nearest tenth

4) For B(x) in Exercise Set 5.1 - 2

   a) Estimate the relative maximum and relative minimum points

   b) Estimate the three zeros

5) For C(x) and D(x) in the Exercise Set of 5.1 - 3 and 5.1 - 4

   a) Estimate the relative maximum and minimum points for C(x)

   b) Estimate the relative maximum and minimum points for D(x)

   c) C(x) = -D(x). Therefore for every x the two functions are additive inverses. Write a general statement regarding the comparison of relative maximum and minimum points for functions such as C(x) and D(x), that are opposites.

6) For E(x) in Exercise Set 5.1 - 5 estimate the four zeros to the nearest tenth.

7) For $F(x)$ and $G(x)$ in Exercise Sets 5.1 - 6 and 5.1 - 7 write both functions in factored form.

(8 - 15) Provide a rough sketch for each of the following functions:

8) $y_1 = 2x^3 + 18x$

9) Relative maximum point (0, 2)
Relative minimum point (-4, -4)
Point of tangency (2, 0)
x- intercepts at -6 and -1

10) 5th degree polynomial $a_5 = 2$
one zero at -1
zero of multiplicity 4 at (2, 0)

11) Turning points (-3, 4), (0, -2) $a_4 < 0$, note turning points refer to relative maximum and relative minimum points. Zeros at -2 and -4. A zero of multiplicity 2 at 1.

12) $y_2 = x(x + 1)(x - 4)^2$

13) $y_3 = -(x - i)(x + i)(x + 1)$

14) $y_4 = -2x^2 \left(x + \frac{1}{2}\right)$

15) $y_5 = x(2x - 3)^2 (x + 1)^2$

5.3 Slopes of Lines Tangent To a Function

In Elementary Algebra you acquired the skill of dividing a polynomial P(x) by a binomial of the form x - c. Obtaining a quotient Q(x) and a remainder R.

EXAMPLE 5.3.1 Given $P(x) = \frac{1}{2}x^3 - 3x^2 + 5x + 4$,

divide P(x) by x - 2

Solution:

$$
\begin{array}{rl}
 & \tfrac{1}{2}x^2 - 2x + 1 \\
\cline{2-2}
x-2 \,\big) & \tfrac{1}{2}x^3 - 3x^2 + 5x + 4 \\
 & \underline{\tfrac{1}{2}x^3 - x^2} \\
 & \qquad -2x^2 + 5x \\
 & \qquad \underline{-2x^2 + 4x} \\
 & \qquad\qquad\quad x + 4 \\
 & \qquad\qquad\quad \underline{x - 2} \\
 & \qquad\qquad\qquad\quad 6 = R
\end{array}
$$

This factoring or division process was shortened in Intermediate Algebra when synthetic division was used

$$\begin{array}{c|cccc} c & a_3 & a_2 & a_1 & a_0 \\ 2 & \frac{1}{2} & -3 & +5 & +4 \\ & & \triangle 1 & \triangle -4 & \triangle 2 \\ \hline & \textcircled{\scriptsize $\frac{1}{2}$} & \textcircled{\scriptsize -2} & \textcircled{\scriptsize 1} & \textcircled{\scriptsize 6} \end{array}$$

If P(x) is written in nested form and the expression is evaluated one step at a time the procedure will be identical to synthetic division.

$$P(x) = \left[\left(\textcircled{\scriptsize $\frac{1}{2}$}x - 3\right)x + 5\right] x + 4$$

for $x = \boxed{2}$

$$\left[\left(\tfrac{1}{2}\boxed{2} - 3\right) x + 5\right]x + 4$$

$$\left[\left(\triangle 1 - 3\right) x + 5\right] x + 4$$

$$\left[\left(\textcircled{\scriptsize -2}\right) x + 5\right] x + 4$$

$$\left[(-2) \boxed{2} + 5\right]x + 4$$

$$\left[\triangle -4 + 5\right] x + 4$$

$$\left[\textcircled{\scriptsize 1}\right] x + 4$$

$$(1) \boxed{2} + 4$$

$$\triangle 2 + 4$$

$$P(2) = R = \textcircled{\scriptsize 6}$$

The method is simply multiplying the leading coefficient by c, adding this product to the next coefficient and continuing the process.

We will now investigate what information division by a binomial will give us.

$$\frac{P(x)}{x-c} = Q(x) + \frac{R}{x-c} \quad X \neq c$$

$$\therefore \quad (i) \quad P(x) = Q(x) \cdot (x-c) + R$$

If $x = c$, $P(c) = Q(c) \cdot (0) + R$ or $P(c) = R$

Repeating this division by $x-c$ on $Q(x)$ results in a second quotient $f(x)$ and a second remainder S.

$$\frac{Q(x)}{x-c} = f(x) + \frac{S}{x-c}$$

$$(ii) \quad Q(x) = f(x) \cdot (x-c) + S$$

Substituting for Q(x) in (i) yields

$$P(x) = [f(x) \cdot (x-c) + S] \quad (x-c) + R$$

$$P(x) = f(x) \cdot (x-c)^2 + (x-c)S + R$$

but $R = P(c)$ as x approaches c.

$$\therefore \quad P(x) = f(x) \cdot (x-c)^2 + (x-c)S + P(c)$$

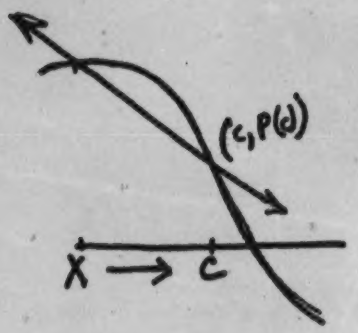
or $$(iii) \frac{P(x) - P(c)}{x-c} = f(x) \cdot (x-c) + S \, *$$

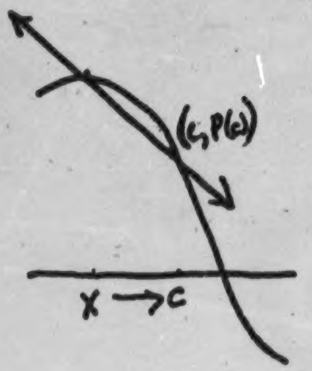
now as $x \longrightarrow c$, the right side of the equation approaches S.

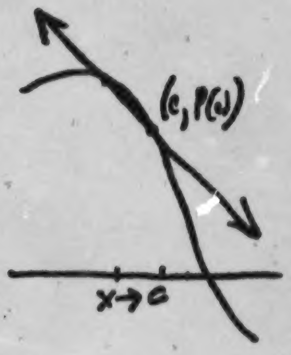
---

* $\frac{P(x) - P(c)}{x-c}$ represents the slope of a line joining the points $(c, P(c))$ and $(x, P(x))$.

Given a function P(x), select a point (c, P(c)). Notice what takes place as $x \longrightarrow c$







The line joining the two points approaches a tangent to the function at (c, P(c)).

The slope of this tangent line is

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{P(x) - P(c)}{x - c}$$

from (iii) as $x \longrightarrow c$ $m \longrightarrow S$ and S is the "second" remainder when P(x) is divided by $x - c$.

From EXAMPLE 5.3.1 $Q(x) = \frac{1}{2}x^2 - 2x + 1$. Dividing by x-2 once again we obtain

$$\begin{array}{r|rrr|l} \underline{2} & \frac{1}{2} & -2 & +1 & \\ & & 1 & -2 & \\ \hline & \frac{1}{2} & -1 & -1 & = S \end{array}$$

Therefore for the function $P(x) = \frac{1}{2}x^3 - 3x^2 + 5x + 4$ when $x = 2$, the line tangent to P(x) at x = 2 has a slope of -1. P(2) = 6, therefore the equation of this tangent line is $y-6 = -1\ (x - 2)$ or $x + y = 8$.

EXAMPLE 5.3.2 Find the equation of the line tangent to $P(x)$ at $x = 4$.

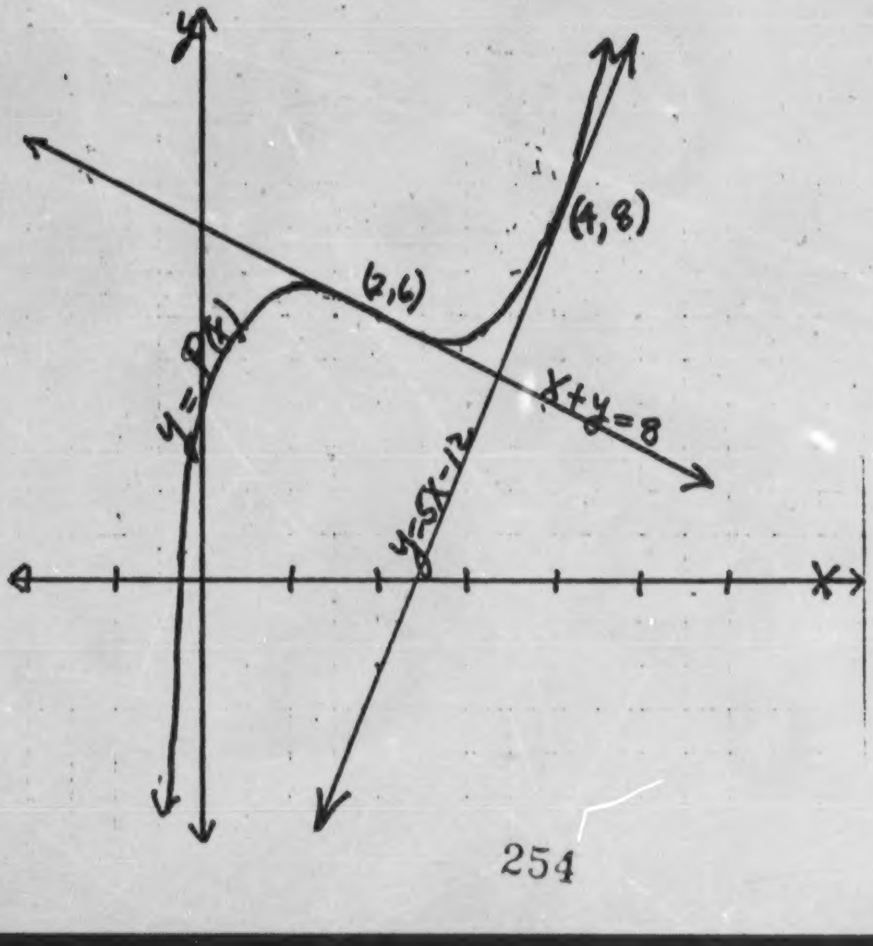
Solution: $P(x) = \frac{1}{2}x^3 - 3x^2 + 5x + 4$ by double synthetic division.

$$\begin{array}{r|rrrl}
\underline{4} & \frac{1}{2} & -3 & 5 & 4 \\
 & & 2 & -4 & 4 \\
\hline
\underline{4} & \frac{1}{2} & -1 & 1 & \vert\ 8 = R = P(4) \\
 & & 2 & 4 & \\
\hline
 & \frac{1}{2} & 1 & \vert\ 5 = S & \\
\end{array}$$

$$y - y_1 = S(x - x_1) \qquad (x_1, y_1) \longleftarrow (4, 8)$$

$$y - 8 = 5(x - 4)$$

$$y = 5x - 12$$



We may now confirm our thoughts regarding the relative maximum and minimum points for the polynomial $P(x) = x^3 + \frac{3}{2}x^2 - 6x - 2$ graphed in Section 5.1.

(-2, 8) seemed to be a relative maximum. By "double division" we obtain

$$\begin{array}{r|rrr|l} -2 & 1 & \frac{3}{2} & -6 & -2 \\ & & -2 & 1 & 10 \\ \hline & 1 & -\frac{1}{2} & -5 & 8 = R \end{array}$$

$$\begin{array}{r|rr|l} -2 & 1 & -\frac{1}{2} & -5 \\ & & -2 & 5 \\ \hline & 1 & -\frac{5}{2} & 0 = S \end{array}$$

$S = 0$. Therefore the slope of the line drawn tangent to $P(x)$ is 0. Only a horizontal line has zero slope, therefore (-2,8) is a turning point, in this case a relative maximum.

Our calculator or computer can assist us in finding the turning points for a function. By writing a program we can search between zeros for these turning points. 'Between every two successive pairs of X-intercepts there is a relative maximum or relative minimum.

"Double Synthetic Division" Algorithm.

Given: $P(x) = a_n x^n + \ldots a_3x^3 + a_2x^2 + a_1x + a_0$

$n \leq 7$

1) Let $x = c$

2) Calculate $P(x)$ by a nested form subroutine, $R = P(x)$. Remember $b_n \ldots b_2$, b_1, and b_0; $n \leq 6$ coefficients of the quotient $Q(x)$ $R_0 \leftarrow b_0$, $R_1 \leftarrow b_1$, etc.

3) *Display R

4) Calculate $Q(x)$, $S = Q(x)$

5) ** Display S, STOP

---

*If $R = 0$, c is a zero of $P(x)$ and $x - c$ is a factor.

** If $S = 0$, (x, R) is a turning point.

Exercise Set 5.3

1) Use double synthetic division to verify that the point (1, -5.5) is a relative minimum for the function $P(x) = x^3 + \frac{3}{2}x^2 - 6x - 2$.

(2 - 7) Write a program and then use it to find all the turning points for the following functions within the interval listed. At most there can be n-1 turning points for a n^{th} degree polynomial. Let d = 1.

2) $A(x) = 2x^3 - 6x^2 - 6x + 3$ $[-1, 3]$

3) $B(x) = x^3 + 3x^2 - 9x + 5$ $[-4, 2]$

4) $C(x) = 2x^3 - 3x^2 - 36x + 10$ $[-3, 4]$

5) $D(x) = x^3 - 3x^2 - 9x + 6$ $[-2, 5]$

6) $E(x) = 3x^4 + 4x^3 - 12x^2 + 3$ $[-3, 2]$

7) $F(x) = x^5 - 5x + 2$ $[-2, 2]$

8) The zeros of $G(x) = -x^3 + x^2 + 2x$ are 0, 2 and -1

   a) Estimate to the nearest tenth the relative maximum point for G(x).

   b) Estimate to the nearest tenth the relative minimum point for G(x).

(9 - 12) Find the equation of the line tangent to the function for the value of x as listed.

9) $f(x) = 4x^3 - 19x - 15$ at $x = -1$

10) $g(x) = x^4 - 2x^3 - 3x^2 - x + 5$ at $x = 2$

11) $h(x) = 2x^3 - x^2 - 22x - 25$ at $x = -1.5$

12) $k(x) = 2x^3 + 3x^2 + 2x - 5$ at $x = .5$

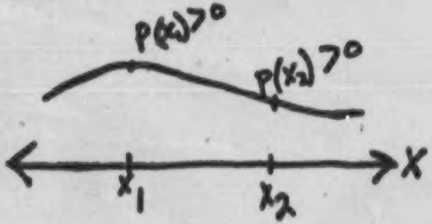
5.4 Calculating Zeros of a Function

In this section we will program the calculator or computer to find the real zeros of a function to any pre-determined degree of accuracy.

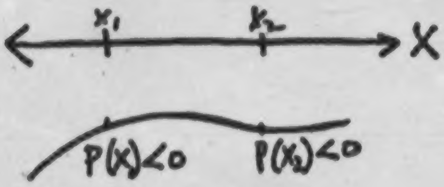
As you may have observed from the graphs of continuous functions in previous sections, the second coordinates for consecutive x values help us in estimating the location of the zeros.

If there is no zero of function P(x) between two values of x, then the function does not intersect the x-axis between these two values and the second coordinates have the same sign.

CASE A



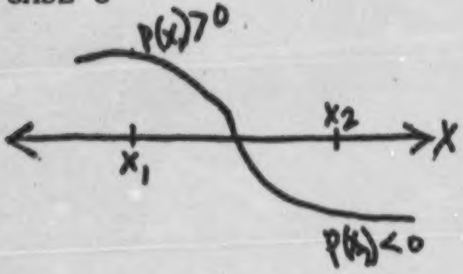
CASE B



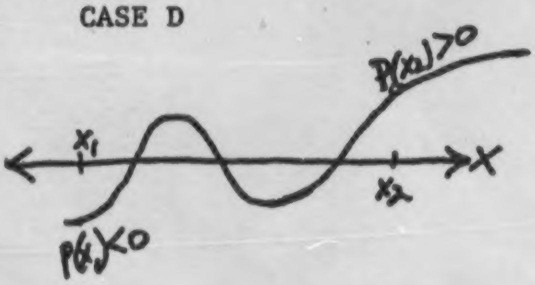
in each case $P(x_1) \cdot P(x_2) > 0$

If a zero is located between two values of x, the graph intersects the x-axis at least once between the two values and

CASE C



CASE D



in each case $P(x_1) \cdot P(x_2) < 0$

EXAMPLE 5.4.1 Given: $P(x) = x^4 - 2x^3 - 4x^2 + 6x + 3$.

Finding the values of $P(x)$ for each x in the interval $[-2, 3]$ with the common difference of .5 yields

Solution:

x	P(x)	
-2	7	
-1.5	-3.2	←
-1	-4	
-.5	-.7	
0	3	←

x	P(x)	
.5	4.8	
1	4	
1.5	1.3	
2	-1.0	←
2.5	.8	←
3	12.0	

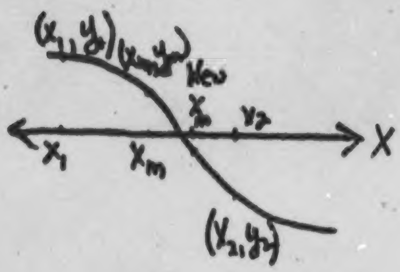
The arrows indicate where the function values change sign. There must be at least one x-intercept between the respective x values for these $P(x)$. Since a fourth degree equation can have no more than four zeros, there will be one zero between -2 and -1.5, one between -.5 and 0, one between 1.5 and 2, and one between 2 and 2.5.

If we sketched the graph of $P(x)$, each zero could probably be estimated to the nearest tenth. It would be quite difficult to be more accurate unless we narrowed our interval and decreased the common difference. We could then test x values until the sign of $P(x)$ changed, repeating the process of "narrowing" the interval and common difference. This is a most time consuming task.

A faster method to obtain a zero of the function would be to find the midpoint of the two x values, $x_m = \frac{x_1 + x_2}{2}$, find $P(x_m)$ and compare this value against one of the known function values.

Let $P(x_1) = y_1$, $P(x_2) = y_2$ and $P(x_m) = y_m$

ILLUSTRATION I



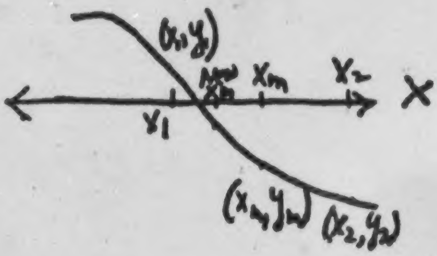
if $y_m \cdot y_2 < 0$

Replace x_1 with x_m

find new midpoint,

evaluate, etc.

ILLUSTRATION II



if $y_m \cdot y_2 > 0$

Replace x_2 with x_m

find new midpoint

evaluate, etc.

Continue this process until the two x values agree to the specified accuracy. This repeating process is called iteration and the method is called the interval bisection method.

Interval Bisection Algorithm

1) Remember x_1, x_2

2) Find $m = \dfrac{x_1 + x_2}{2}$

3) Find $P(m)$

4) Find $P(x_2)$

5) If $P(m) \cdot P(x_2) > 0$, $x_2 \leftarrow m$, Display x_1, x_2 } go to STEP 8

6) If $p(m) \cdot P(x_2) < 0$, $x_1 \leftarrow m$, Display x_1, x_2 }

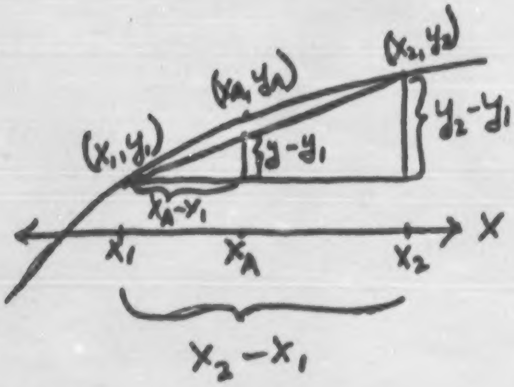
7) If $P(m) \cdot P(x_2) = 0$, DISPLAY m, STOP

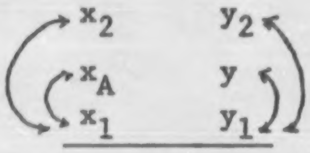
8) Compare the new values for x_1 and x_2. If they agree to the required number of decimal places, STOP.

9) GO TO STEP 2

This interval bisection algorithm when used to find the zero of P(x) between 2 and 2.5 in EXAMPLE 5.4.1 requires 12 iterations to obtain an accuracy to the nearest ten thousandth.

A method that will be much faster than interval bisection is based on the familiar linear interpolation approach you used in Intermediate Algebra. When interpolating, two points (x_1, y_1) and (x_2, y_2) are selected on a continuous function. If these two points are picked close enough together, a line segment drawn between them is not very distinguishable from the actual curved line of the function. By using equal ratios from similar triangles an interpolated y value is computed that is close to the actual value y_A on the curve. For the illustration the two points are selected quite far apart

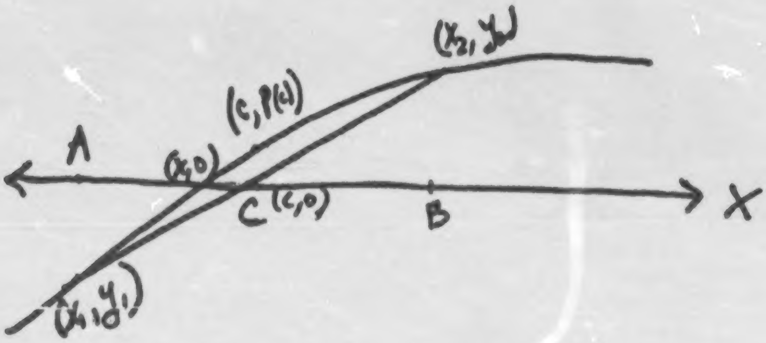




$$\frac{x_A - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

This same straightness assumption is used when finding the zero of a function. By a recursive process the error made in estimating is successively "narrowed down".

To find this x value where P(x) = 0, we begin by selecting two points A and B such that their second coordinates are opposite in sign. If the function is continuous, this insures a x-intercept between A and B. By assuming the curve is a straight line, we obtain an estimate c for the zero of the function.



We compute the value of c by using slopes.

$$\frac{0 - y_1}{c - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

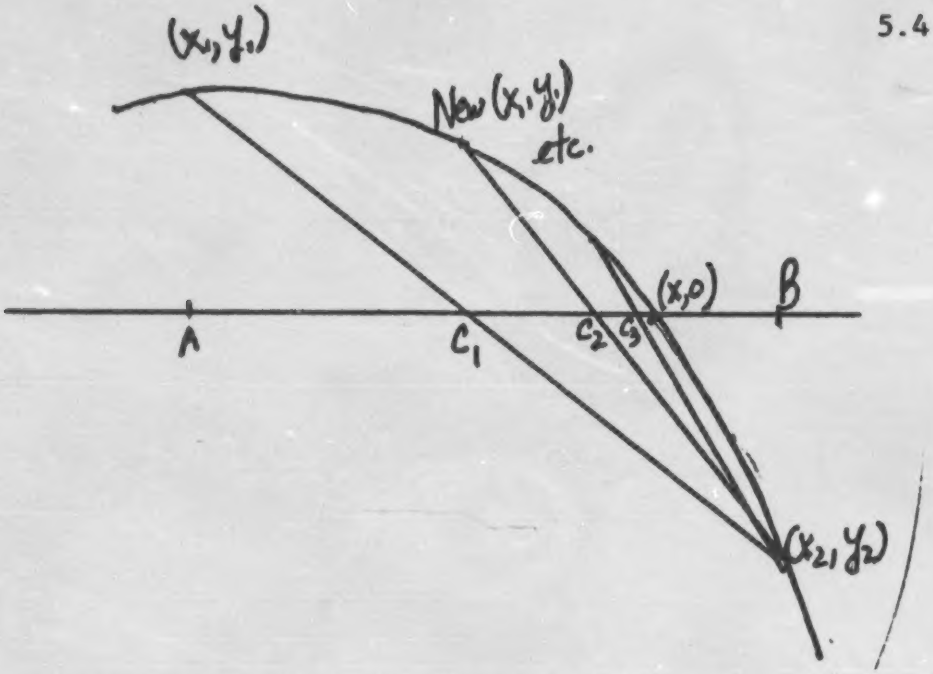
$$(c - x_1)(y_2 - y_1) = -y_1 (x_2 - x_1)$$

$$c - x_1 = \frac{x_1 y_1 - x_2 y_1}{y_2 - y_1}$$

$$c = \frac{x_1 (y_2 - y_1)}{y_2 - y_1} + \frac{x_1 y_1 - x_2 y_1}{y_2 - y_1}$$

$$c = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1}$$

We then compute P(c) and continue the method using point (c, P(c)) with (x_1, y_1) or (x_2, y_2) depending on which points second coordinate has the opposite sign as P(c). As we continue the process each line segment crosses closer to the point where P(x) = 0, the desired zero of the function.



Repeated Linear Interpolation Algorithm.

```
1) Remember $x_1$ and $x_2$
2) Calculate $y_1$ and $y_2$
3) Find $c = \frac{x_1 \cdot y_2 - x_2 y_1}{y_2 - y_1}$
       (To save steps the $\sum +$ and $\sum -$ keys may
        be helpful, when using a calculator.)
4) Calculate $P(c)$, IF $P(c) = 0$   STOP
5) If $P(c) \cdot y_1 < 0$   GO TO STEP 7
6) If $P(c) \cdot y_1 > 0$   GO TO STEP 8
7) $(x_2, y_2) \longleftarrow (c, P(c))$,  display $x_2$   GO TO STEP 9
8) $(x_1, y_1) \longleftarrow (c, P(c))$,  display $x_1$   GO TO STEP 9
9) When $x_1$ or $x_2$ does not change the value to the decimal
   accuracy required  STOP  otherwise GO TO STEP 2.
```

Exercise Set 5.4

1) Draw four separate illustrations, two of increasing and two of decreasing functions, such that in two illustrations (c, P(c)) would continually replace (x_1, y_1) and in two illustrations (c, P(c)) continually replaces (x_2, y_2).

2) Use the linear interpolation algorithm to find the zero of EXAMPLE 5.4.1 between 2 and 2.5 to the nearest ten thousandth. Count the number of iterations required.

3) Find the remaining three zeros of EXAMPLE 5.4.1 to four decimal place accuracy.

(4 - 7) Find all the real zeros of the following functions to four place accuracy:

4) $A(x) = x^3 - 3x + 1$

5) $B(x) = 20x^3 - 30x^2 + 12x - 1$

6) $C(x) = x^4 - 3x^3 - 20x^2 - 24x - 8$

7) $D(x) = x^4 + 5x^3 - 27x^2 + 31x - 10$

8) $E(x) = x^3 + 4x^2 + 10x + 15$

9) $F(x) = x^3 + 2x^2 - 7x + 1$

5.5 Factoring Polynomial Functions

In Section 5.2, when possible, polynomial functions were written in factored form. To obtain a factored polynomial expression we make use of the remainder and factor theorems used in Section 5.3.

$$\frac{P(x)}{x - c} = Q(x) + \frac{R}{x-c}$$

$$P(x) = Q(x)\ (x-c) + R$$

$$\text{if} \quad x = c$$

$$P(c) = Q(c)\ (0) + R$$

$P(c) = R$ **Remainder Theorem**

If $R = 0$, $P(c) = 0$ and $x-c$ is a factor of $P(x)$ **Factor Theorem**

To find the remainder, substitute values for x in P(x). If the result is zero, obtain the remaining polynomial after factoring by the synthetic process illustrated in 5.3.

If the polynomial has rational factors, c will be a rational number. Only certain values for c need to be tested.

Rational Zero Theorem

Given $P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots a_2x^2 + a_1x + a_0$ $a_n \dots a_1, a_0$ being rationals. If P(x) has a rational zero, its numerator is a factor of a_0 and its denominator is a factor of a_n.

Proof: Assume this rational zero is in simplest form $\frac{p}{q}$, p and q are relatively prime integers having no common factor.

Since $\frac{p}{q}$ is a zero

$$p(\tfrac{p}{q}) = a_n(\tfrac{p}{q})^n + a_{n-1}(\tfrac{p}{q})^{n-1} + \dots a_2(\tfrac{p}{q})^2 + a_1(\tfrac{p}{q}) + a_0 = 0$$

multiplying by q_n we have

$$(1) \quad a_n p^n + a_{n-1} p^{n-1} q + \dots a_2 p^2 q^{n-2} + a_1 p^1 q^{n-1} + a_0 q^n = 0$$

or

$$a_{n-1} p^{n-1} q + \dots a_2 p^2 q^{n-2} + a_1 p q^{n-1} + a_0 q^n = -a_n p^n$$

q is a factor of each term on the left side, therefore q must be an exact factor of the right side of the equation. p and q are relatively prime, the only conclusion is that q is a factor of a_n.

Similarly from (1)

$$a_n p^n + a_{n-1} p^{n-1} q + \dots a_2 p^2 q^{n-2} + a_1 p q^{n-1} = -a_0 q^n$$

p is an exact factor of a_0

EXAMPLE 5.5.1 Factor the polynomial

$$p(x) = 4x^4 + 12x^3 - 15x^2 - 38x + 30$$

Solution: From the Rational Zero Theorem, if $\frac{p}{q}$ is a zero p is a factor of 30; ± (1, 2, 3, 5, 6, 10, 15, 30) and q is a factor of 4; ± (1, 2, 4).

Possible rational zeros $\pm\left(\frac{1,\ 2,\ 3,\ 5,\ 6,\ 10,\ 15,\ 30}{1,\ 2,\ 4}\right)$

which are the integers $\pm$ (1, 2, 3, 5, 6, 10, 15, 30) and the fractions $\pm$ (¼, ½, 3/4, 5/4, 3/2, 5/2, 15/2, 15/4)

Rewriting the polynomial in nested form and testing the integers by program, yields

$$P(x) = x\,(x(x(4x + 12) - 15) - 38) + 30$$

x	P(x)
-6	2370
-5	845
-3	9
-2	14
-1	45

x	P(x)
0	30 ←
1	-7 ←
2	54
3	429

No need to continue as P(x) will continue to increase.

There is a zero between 0 and 1. The only possible rational zeros here are ¼, ½, 3/4

x	P(x)
.25	19.8
.5	9 ←
.75	-.6

No rational zeros from 0 to 1.

There is a zero between 1 and 2. The only possible rational zeros are 5/4 and 3/2

x	P(x)
1.25	-7.7
1.5	0

P(1.5) = 0. Therefore x - 3/2 is a factor.

By synthetic division

3/2	4	12	-15	-38	30
		6	27	18	-30
	4	18	12	-20	0

$Q(x) = 4x^3 + 18x^2 + 12x - 20$

$P(x) = (x-c) \cdot Q(x) + R$
$P(x) = (x - 3/2)(4x^3 + 18x^2 + 12x - 20) + 0$
$P(x) = (2x - 3)(2x^3 + 9x^2 + 6x - 10)$

Any factor of Q(x) will also be a factor of P(x). There is no need to retest values that did not work for P(x).

We will now search for any remaining rational zeros. One possibility is that x-3/2 is a factor more than once (Graph is tangent to x-axis)

$$Q(x) = x\left[x(2x + 9) + 6\right] - 10$$

x	Q(x)
1.5	26

Other possibilities: Between any two x values the graph of the function could have intersected the x-axis an even number of times. The sign of the f(x) values would not change. If the sign changes there could be an odd number of intersection points.

Inspecting within the interval $[-3, -2]$ we obtain another zero

x	Q(x)
-2.5	0

-5/2	2	9	6	-10
		-5	-10	+10
	2	4	-4	0

$$P(x) = (2x - 3)(x + \frac{5}{2})(2x^2 + 4x - 4)$$
$$P(x) = (2x - 3)(2x + 5)(x^2 + 2x - 2)$$

Once we have reduced the polynomial to an expression where a quadratic is remaining, the final two zeros may be obtained, if necessary, by the quadratic formula.

$$x^2 + 2x - 2 = 0$$
$$x = -1 + \sqrt{3} \doteq .73$$
$$x = -1 - \sqrt{3} \doteq -2.73$$

Final factored form

$$P(x) = (2x-3)(2x+5)(x + 1 - \sqrt{3})(x + 1 + \sqrt{3})$$

Two rational zeros $3/2, -5/2$

Two irrational zeros $-1 + \sqrt{3}, \ -1 - \sqrt{3}$

Irrational zeros of the form $a + \sqrt{b}$ will always occur as solutions in conjugate pairs $a + \sqrt{b}, \ a - \sqrt{b}$. As mentioned in Section 5.2, this is also true for zeros that are not real. Complex zeros will occur in conjugate pairs $a + bi, a - bi$.

Exercise Set 5.5

(1 - 12) Rewrite the following polynomials in factored form.

1) $A(x) = 6x^3 + 11x^2 - x - 6$

2) $B(x) = 2x^3 - 5x^2 - 9x + 18$

3) $C(x) = 2x^4 + 11x^3 + 9x^2 - 27x - 27$

4) $D(x) = 2x^4 - 11x^3 + 21x^2 - 16x + 4$

5) $E(x) = x^3 - 2x^2 + 4x - 8$

6) $F(x) = 3x^3 - 5x^2 + 3x - 5$

7) $G(x) = 3x^4 - x^3 - 7x^2 + 4x - 20$

8) $H(x) = 2x^4 - x^3 - 15x^2 + 9x - 27$

9) $J(x) = 8x^4 - 36x^3 + 54x^2 - 27x$

10) $K(x) = 45x^4 - 42x^3 - 4x^2 + 8x$

5.6 Rational Functions

A rational function is a quotient of two polynomial functions

$$R(x) = \frac{f(x)}{g(x)} \quad g(x) \neq 0$$

Rational functions are similar to rational numbers and may be expressed in various ways

$$\frac{4}{7} = \frac{8}{14} = \frac{12}{21} = \frac{-40}{-70} = \ldots$$

$$\frac{x}{x^2+x} = \frac{1}{x+1} = \frac{x^2+1}{(x+1)(x^2+1)} = \frac{-2}{-2x-2} = \ldots$$

The simplest rational expression is when the numerator and denominator have no common factors. The expression is then in lowest terms.

For the rational function above $\frac{1}{x+1}$ is in lowest terms. The graph of this rational function however, will differ from the graph of $\frac{x}{x^2+x}$ because of different domains.

The domain of a rational function $R(x) = \frac{f(x)}{g(x)}$ consists of all real numbers except for the real zeros of $g(x)$. (This last requirement is to insure that $R(x)$ is defined; that is, the denominator is not zero.)

EXAMPLE 5.6.1 The domain of $\frac{1}{x+1}$ consists of all real numbers $x \neq -1$.

The domain of $\frac{x}{x^2 + x}$ is any real number x, $x \neq 0$ or $x \neq -1$.

For some disallowed replacements of the variable in a rational function, both the numerator and denominator equal zero. For other replacements only the denominator equals zero.

EXAMPLE 5.6.2 $S(x) = \frac{f(x)}{g(x)} = \frac{x^2 - 9}{x - 3}$

Solution: If $x = 3$, $f(x) = 0$ and $g(x) = 0$

$T(x) = \frac{f(x)}{g(x)} = \frac{2x + 1}{x - 2}$

if $x = 2$, only $g(x) = 0$

If a rational function $R(x) = \frac{f(x)}{g(x)}$ has a value for $x = a$ such that $g(a) = 0$ and $f(a) \neq 0$, then the function $R(x)$ has a vertical asymptote* at $x = a$.

If a rational function $R(x) = \frac{f(x)}{g(x)}$ has a value for $x = a$ such that both $g(a) = 0$ and $f(a) = 0$, then the function will have a "missing point" on the graph where $x = a$.

---

* An asymptote is a line toward which a graph tends as x "gets close" or approaches some specific value or as x increases or decreases without bound.

For the example $T(x) = \frac{2x + 1}{x - 2}$

as $x \longrightarrow 2$ from either direction only $g(x) \longrightarrow 0$ and $|T(x)| \longrightarrow \infty$.

x	T(x)
1.99999	-499,998
2.00001	+500,002

x = 2 will be a vertical asymptote

For $S(x) = \frac{x^2 - 9}{x - 3}$

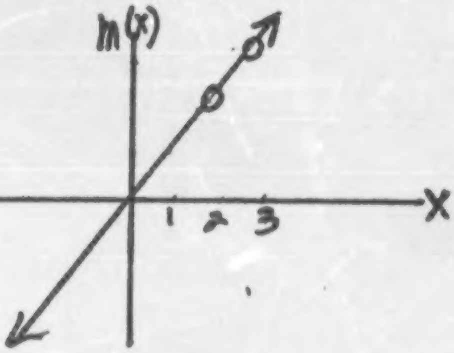
as $x \longrightarrow 3$ both f(x) and $g(x) \longrightarrow 0$ and $S(x) \longrightarrow k$

x	S(x)
2.99999	6.00000
3.00001	6.00000

at x = 3, there will be a missing point

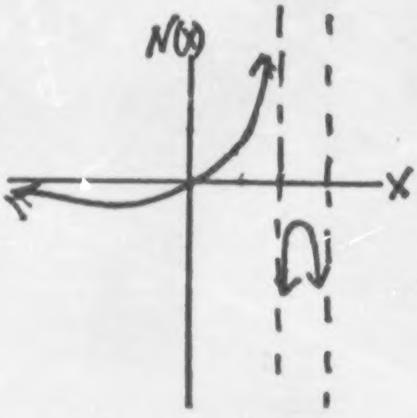
EXAMPLE 5.6.3 $\quad M(x) = \frac{x^3 - 5x^2 + 6x}{(x-3)(x-2)}$

M(x) will have missing points at x = 3 and x = 2



EXAMPLE 5.6.4 $\quad N(x) = \frac{x}{(x-3)(x-2)}$

vertical asymptotes at x = 3 and x = 2.



If the numerator of a rational function is one degree more than the degree of the denominator, the function may contain a *"slant" asymptote.

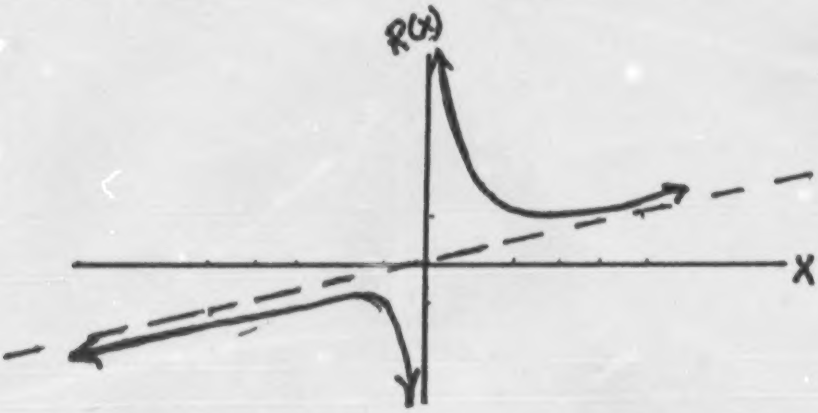
EXAMPLE 5.6.5 $R(x) = \frac{x^2 + 4}{5x}$

Solution: By division $R(x) = \frac{1}{5}x + \frac{4}{5x}$

as $|x| \longrightarrow \infty$ $\frac{4}{5x} \longrightarrow 0$ and $R(x) \longrightarrow \frac{1}{5}x$

$R(x) = \frac{1}{5}x$ is the slant asymptote.

$x = 0$ is a vertical asymptote.



If the degree of the numerator of a rational function is less than or equal to the degree of the denominator, the rational function will have a horizontal asymptote:

---

*A slant asymptote is an oblique straight line.

EXAMPLE 5.6.6 f(x) and g(x) same degree

$$S(x) = \frac{3x}{x + 1}$$

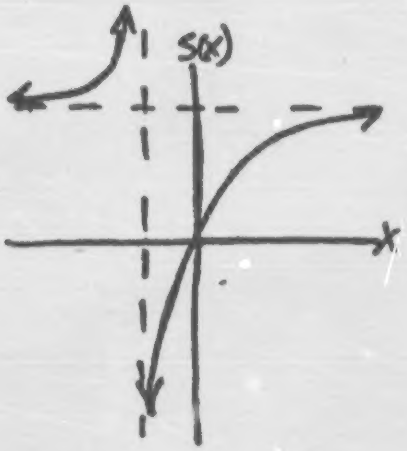
By division

$$S(x) = 3 + \frac{3}{x+1}$$

as $|x| \rightarrow \infty$ $S(x) \rightarrow 3$

$S(x) = 3$, horizontal asymptote

$x = -1$, vertical asymptote



EXAMPLE 5.6.7 deg f(x) deg g(x)

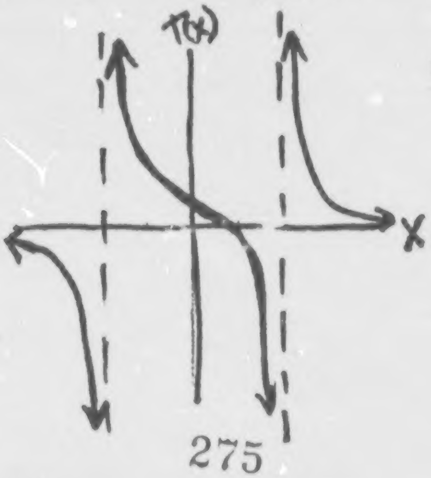
Solution: $T(x) = \frac{x-1}{x^2-4}$ $x = 2$ and $x = -2$ are vertical asymptotes.

Divide each term by the highest degree variable.

$$T(x) = \frac{\frac{1}{x} - \frac{1}{x^2}}{1 - \frac{4}{x^2}}$$

as $|x| \longrightarrow \infty$ $T(x) \longrightarrow 0$

$T(x) = 0$ is a horizontal asymptote.



Exercise Set 5.6

1) Write a program to assist you in graphing $F(x) = \frac{-8x + 30}{2x - 3}$, for an interval of $[-8, 12]$ $d = 1$. What difficulty arises when d is changed to .5?

(2 - 4) Over the years, many functions have acquired special names. Investigate the following functions, determine when possible the domain, range and asymptotes. Use the algorithms from previous sections to graph each function.

2) The "serpentine" $S(x) = \frac{-ax}{x^2 + b^2}$

Let $a = -6$, $b = 2$, Interval $[-5, 5]$

3) The "pilaster" $P(x) = \frac{-ax}{x^2 - b^2}$

for $a = 1$, $b = 2$, Interval $[-5, 5]$

4) The "witch" $W(x) = \frac{a^3}{x^2 + a^2}$

for $a = 2$ Interval $[-5, 5]$

(5 - 12) For the following Rational functions, use the algorithms to assist you in sketching the graphs within the interval listed.

5) $M(x) = \frac{x(x + 1)}{(x+1)(x-1)}$ $[-3, 3]$

6) $T(x) = \frac{x^2(x - 1)}{(x+1)^3(3x-2)^2}$ $[-4, 4]$

7) $U(x) = \frac{3x^2 (x + 1)}{2(x-2)^2(x-1)}$ $[-3, 3]$

8) $V(x) = \frac{(x - 3)^2}{x^3 - 27}$ $[-4, 4]$

9) $N(x) = \frac{3x^3 - x^2 + 1}{x^2 + 1}$ $[-3, 3]$

10) $A(x) = \frac{2x^3}{x^3 - 1}$ $[-4, 4]$

11) $B(x) = \frac{x^2(x - 2)}{(x-1)^3}$ $[-4, 4]$

12) $C(x) = \frac{x(x - 2)}{x + 1}$ $[-4, 4]$

13) Given $y = \frac{x^2}{x^2 - x+1}$

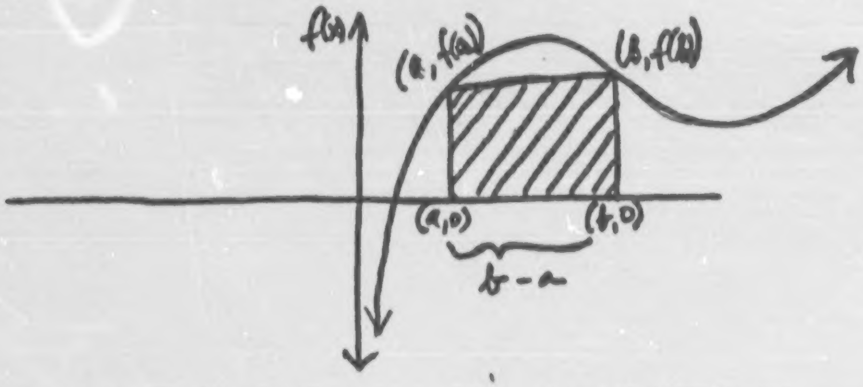
Sketch the function. Using various intervals, determine the highest and lowest value of y. (The range of the function)

14) Determine the range of the function $y = \frac{x^2}{x^2 - x+1}$ using an algebraic approach

5. 7 Area

Many times in mathematics you have been required to find the area of circular, rectangula, square, etc. regions. In this section you will be expected to evaluate the area of irregular shaped regions. Problems concerning areas, volumes, masses and other mathematical and physical topics will be in abundance during future courses. In the calculus this will involve a method called integration. Until that time, however, we will use the calculator and the formula for the area of a trapezoid to solve these problems.

Consider an irregular shaped region bounded by a continuous function f(x), the x-axis and two vertical lines x = a and x = b.

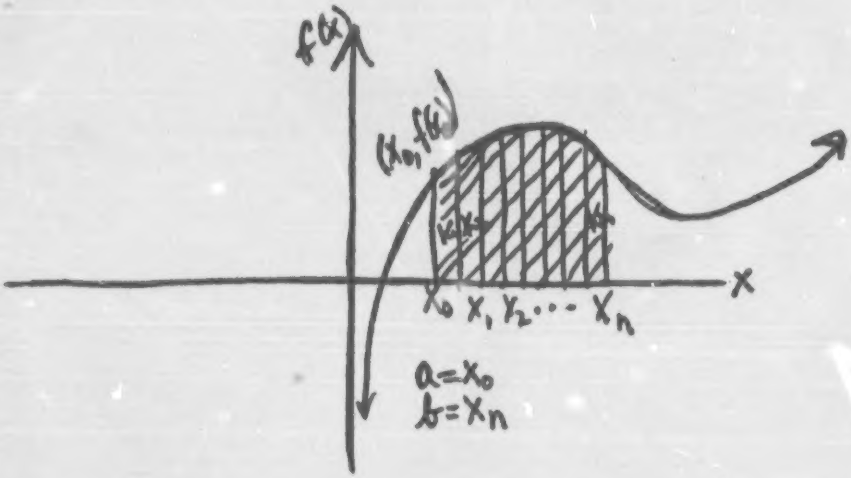


If we wish to find the area of region S (shaded), we may form a trapezoid and evaluate its area.

$$K = \frac{1}{2}h\ (b_1 + b_2)$$

$$K = \frac{1}{2}(b - a)(f(a) + f(b))$$

This area k will approximate the required area S. The approximation is not too near the area of S. However, if we divide the interval $[a, b]$ into n smaller congruent intervals, n trapezoids can be formed and the sum of the areas of these n trapezoids will be much closer to the area S.



As we increase the number of trapezoids, the area may be computed to a finer measurement of accuracy.

$$S = k_1 + k_2 + k_3 + \ldots k_n$$

$$k_1 = \tfrac{1}{2}(x_1 - x_0)\,\bigl[f(x_0) + f(x_1)\bigr]$$

$$k_2 = \tfrac{1}{2}(x_2 - x_1)\,\bigl[f(x_1) + f(x_2)\bigr]$$

$$\vdots$$

$$k_{n-1} = \tfrac{1}{2}(x_{n-1} - x_{n-2})\,,\,\bigl[f(x_{n-2}) + f(x_{n-2})\bigr]$$

$$k_n = \tfrac{1}{2}(x_n - x_{n-1})\,\bigl[f(x_{n-1}) + f(x_n)\bigr]$$

$$\therefore\quad S = \tfrac{1}{2}\left(\frac{x_n - x_0}{n}\right)\bigl[f(x_0) + 2\cdot f(x_1) + 2\cdot f(x_2) + \ldots 2f(x_{n-1}) + f(x_n)$$

OR

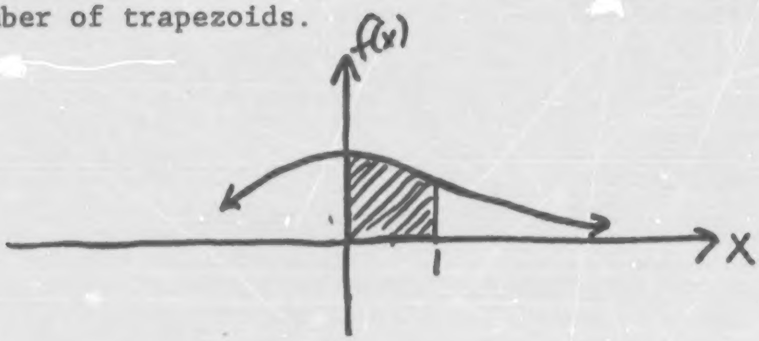
$$S = \frac{(x_n - x_0)}{n}\,\bigl[\tfrac{1}{2}\,(f(x_0) + f(x_n)) + f(x_1) + f(x_2) + \ldots f(x_{n-1})\bigr]$$

Area by Trapezoids Algorithm

```
 1)  Remember A, B, N
 2)  Let $H = \frac{B-A}{N}$ , Remember H
 3)  Calculate  f(A), f(B)
 4)  Let $C = \frac{f(A) + f(B)}{2}$ , Remember  $|c|$ *
 5)  Let N = N-1, if N = 0, Go to 9
 6)  Let A = A + H
 7)  Calculate  $|f(A)|$ *
 8)  C = C + f(A), Go to 5
 9)  S = H · C
10)  STOP
```

EXAMPLE 5.7.1 Given $f(x) = \frac{1}{1 + x^2}$ find the area under the curve to five decimal places for the interval $[0, 1]$.

N = number of trapezoids.



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[*] Absolute value symbols may be used when the area of the entire region being calculated is "below" the x-axis. Otherwise the area will be represented as a negative number.

for N = 5 $S \doteq .78373$

for N = 10 $S \doteq .78498$

for N = 20 $S \doteq .78529$

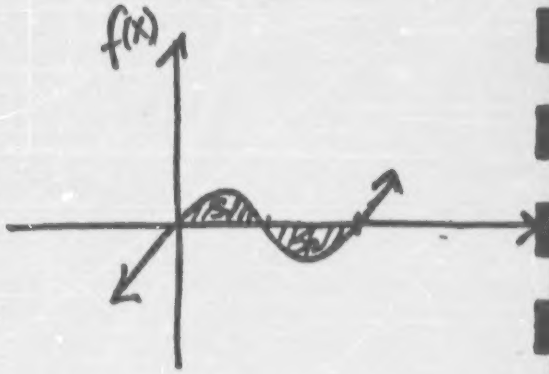
for N = 50 $S \doteq .78538$

for N = 100 $S \doteq .78539$

The exact value using Calculus is $\frac{\pi}{4} \doteq .785398$.

A function may "dip" below the x-axis. Care must therefore be taken that a zero of the function is not located in the interior of one of the n trapezoids. Inaccurate results would then be attained. One approach that will help avoid difficulty is to compute the area above and the area below the x-axis separately. Then add these areas together.

EXAMPLE 5.7.2 $f(x) = x^3 - 3x^2 + 2x$



Solution:

Find to the nearest hundredth the area of the region "between" the x-axis and the function for the interval $[0, 2]$. By factoring $f(x) = x(x-2)(x-1)$ zeros are at 0, 1, 2

$$\left.\begin{array}{ll} \text{For } N = 50 & S_1 \doteq .250 \\ \text{For } N = 50 & S_2 \doteq .250 \end{array}\right\} \text{ symmetrical regions}$$

Total Area .50

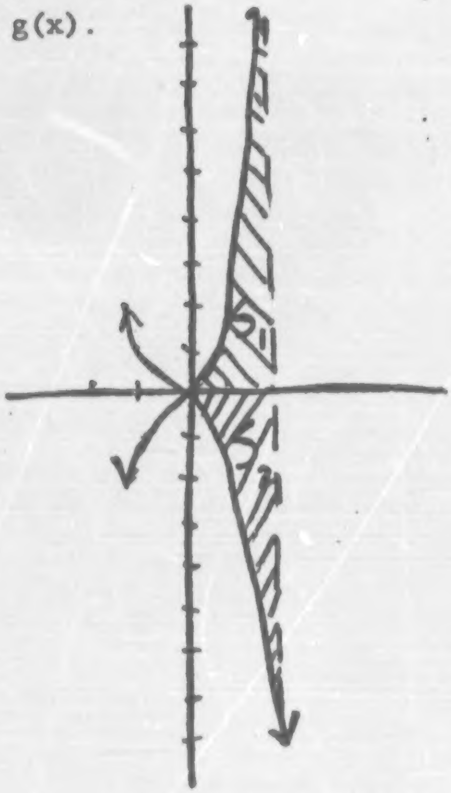
This same method may be employed when finding the area between two continuous functions $f(x) \geq g(x)$.

EXAMPLE 5.7.3 Find the area to the nearest tenth for the region between $f(x) = x^3$ and $g(x) = -x^3$, interval $[0, 2]$.

For N = 50 $s_1 = 4.00$

For N = 50 $s_2 = 4.00$

Total Area = 8.0



Exercise Set 5.7

(1 - 7) Find the area of between the graph of the function and the x-axis within the interval listed.

1) $f(x) = x^3$ $[0, 4]$

2) $g(x) = x^3 - 4x$ $[-2, 2]$

3) $h(x) = x^3 + 12x + 4$ $[1, 3]$

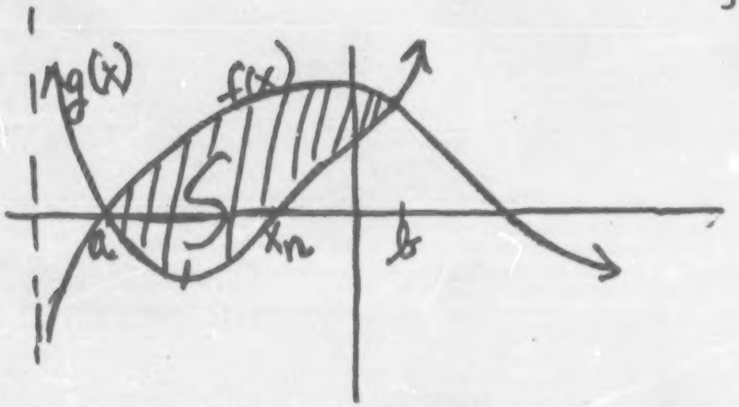
4) $j(x) = (x + 1)^3$ $[0, 1]$

5) $k(x) = x^6 - x^2$ $[0, 1]$

6) $\ell(x) = 2x^2 + 2x^8$ $[-3, 0]$

7) $m(x) = 2x^3 - 5x^2 + 2x$ $[0, 2]$

8)



Given f(x) and g(x) as graphed, explain a procedure to find the area of the shaded region between the two curves. The x-values of the points of intersection are a and b.

(9-10) Using a method similar to problem #8 above, calculate the area of the region enclosed by the curves.

9) $A(x) = \frac{3}{2}x + 1 \quad B(x) = (x-1)^2$

10) $f(x) = 9 - x^2 \quad g(x) = x^2(x^2 - 9)$

5.8 Chapter 5 Test

(1 - 6) Multiple choice

1) Which of the following is a polynomial function?

(1) $f(x) = 2x^{2/3} + x + 1$ (2) $f(x) = \frac{x+1}{x+2}$

(3) $f(x) = x^2 + 3x - 7$ (4) $f(x) = x^2$

2) Which of the following is a continuous function?

(1) $f(x) = \tan x$ (2) $f(x) = x^{37} + x^{14} - 17$

(3) $f(x) = \frac{x+1}{x+2}$ (4) $f(x) = x^x$

3) Which of the following functions has exactly three zeros?

(1) $f(x) = 3x^2 + 3x + 3$ (2) $f(x) = 3 \sin x$

(3) $f(x) = 3^x$ (4) $f(x) = 3x^3 - 3$

4) Which of the following functions is tangent to the x-axis at $x = 2$?

(1) $f(x) = (x-2)(x+1)(x+3)$ (2) $f(x) = 2^x$

(3) $f(x) = (x-2)^2(x+1)$ (4) $f(x) = 2 \ln x$

5) Which of the following functions has a "hole" in its graph at $x = 2$?

(1) $f(x) = \frac{(x+1)(x-2)}{x-2}$ (2) $f(x) = \frac{x+1}{x-2}$

(3) $f(x) = (x-2)^2$ (4) $f(x) = x-2$

6) Which of the following has an asymptote?

(1) $f(x) = x^3 + 2x + 1$ (2) $f(x) = \frac{x+1}{x+2}$

(3) $f(x) = \sin x$ (4) $f(x) = \ln x$

7) Each of the following refer to the polynomial:

$$f(x) = (x+2)(x-1)(x+.5)^2$$

a) What are the zeros of $f(x)$?

b) Make a rough sketch of $f(x)$.

c) Write $f(x)$ as a polynomial in standard form.

d) What is the equation of the line that is tangent to $f(x)$ at $x = -1$?

e) Determine an interval that contains the relative minima of $f(x)$.

8) (a - b) Refer to the following polynomial:

$$f(x) = 9x^5 - 15x^4 + 34x^3 + 26x^2 - 27x + 5$$

a) Determine all possible rational roots for $f(x)$.

b) Determine the rational roots of $f(x)$.

9) a) Sketch the graphs of the following two functions:

$$f(x) = 2-x^2$$

$$g(x) = -x$$

b) Find the points that satisfy the two equations simultaneously.

c) Find the area bounded by the two functions (nearest hundredth). Use N = 50.

CHAPTER 6. EXPONENTS AND LOGARITHMS

In this chapter your ideas about exponents and logarithms will be extended.

6.1 From Rational to Real Exponents

In your earlier study of mathematics, positive integral exponents were characterized by the following definition:

Definition 6.1.1 $b^n = \underbrace{b \cdot b \cdot b \quad \dots \quad \cdot b}_{n \text{ factors}}$, n a positive integer.

You can easily calculate exact answers for expressions like 2^8 or 5^7 by either using the $\boxed{y^x}$ key, a $y \uparrow x$ command or repeated multiplications. When numbers become larger than can be exhibited on the display they are written in scientific notation and the exact value is not immediately available.

Thus $7^{21} = 5.5855 \quad 10^{17}$.

Clever algebraic techniques can be employed to remedy this situation.

EXAMPLE 6.1.2 Determine 7^{21} exactly.

$$\begin{aligned}
7^{21} &= (7^7)^3 = (823543)^3 = (823 \times 10^3 + 543)^3 \\
&= (823 \times 10^3)^3 + 3(823 \times 10^3)^2(543) + 3(823 \times 10^3)(543)^2 \\
&\quad + 543^3 \\
&= (557{,}441{,}767 \times 10^9) + (1{,}103{,}368{,}941 \times 10^6) \\
&\quad + 729{,}982{,}181 \times 10^3 + 160{,}103{,}007 \\
&= \begin{array}{r} 557{,}441{,}767{,}000{,}000{,}000 \\ 1{,}103{,}368{,}941{,}000{,}000 \\ 729{,}982{,}181{,}000 \\ \underline{160{,}103{,}007} \end{array} \\
&= 558{,}545{,}866{,}083{,}284{,}007
\end{aligned}$$

EXAMPLE 6.1.3 Determine 3^{25} exactly.

$$\begin{aligned}
3^{25} &= 3^{18} \cdot 3^7 = 387{,}420{,}489 \; [2187] \\
&= 387{,}420{,}489 \; [2 \times 10^3 + 1 \times 10^2 + 8 \times 10^1 + 7] \\
&= \begin{array}{r} 774{,}840{,}978{,}000 \\ 38{,}742{,}048{,}900 \\ 30{,}993{,}639{,}120 \\ \underline{2{,}711{,}943{,}423} \end{array} \\
&= 847{,}288{,}609{,}543
\end{aligned}$$

Notice that each of the examples made use of exponents and scientific notation. EXAMPLE 6.1.1 used the identity $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$. EXAMPLE 6.1.2 used the distributive property. Even the most sophisticated computers will only display a predetermined number of digits so such techniques are often necessary.

Rational exponents have meaning when we define

Definition 6.1.4

$$b^{x/y} = \sqrt[y]{b^x},\ b > 0,\ x \text{ and } y \text{ are integers and } \frac{x}{y} \text{ is reduced.}$$

$$b^{x/y} = \sqrt[y]{b^x},\ b < 0,\ y \text{ odd, } x \text{ and } y \text{ are integers and } \frac{x}{y} \text{ is reduced.}$$

We must be careful to notice that even roots of positive numbers designate principal roots so $\sqrt{4}$ equals 2, not ± 2.

Recall also that even roots of negative numbers are imaginary so $\sqrt{-4} = 2i$. Most calculators cannot even process odd roots of negative numbers so we need to exercise a little good mathematical sense.

EXAMPLE 6.1.5 $\sqrt[3]{-27}$ yields an error message but $\sqrt[3]{27} = 3$.

Since odd roots of negative numbers are negative $\sqrt[3]{-27} = -3$.

EXAMPLE 6.1.6 $\sqrt{-10}$ yields an error message but $\sqrt{10} \doteq 3.1623$.

Since even roots of negative numbers are imaginary $\sqrt{-10} \doteq 3.1623i$.

It is important to remember that your display is always a rational number (because it terminates) so often you are determining a rational approximation of an irrational number. By definition, irrational numbers are numbers that cannot be expressed as terminating or repeating decimals.

Real exponents have meaning when we accept the following very important statement:

> Every real number can be expressed as the sum of an infinite series of rational numbers.

The proof of this statement is not within the realm of this course, but you have seen these ideas before in Chapter 2. You will see further consequences of this statement and a proof in future mathematics courses.

EXAMPLE 6.1.7

2 is the limit of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

3.28 is the limit of series $3 + .2 + .08 + .000 + .0000 + \ldots$

$\frac{5}{9}$ is the limit of the series $.5 + .05 + .005 + .0005 + \ldots$

$\sqrt{2}$ is the limit of the series $1 + .4 + .01 + .004 + .0002 + \ldots$

Now we can view real exponents in this new setting.

EXAMPLE 6.1.8

$$8^2 = 8^{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots} = 8^1 \cdot 8^{\frac{1}{2}} \cdot 8^{\frac{1}{4}} \cdot 8^{\frac{1}{8}} \cdot 8^{\frac{1}{16}} \ldots$$

$$= 8 \cdot \sqrt{8} \ \sqrt[4]{8} \ \sqrt[8]{8} \ \sqrt[16]{8} \ \ldots$$

$$4^{3.28} = 4^{3+.2+.08+.000+\dots} = 4^3 \cdot 4^{.2} \cdot 4^{.08} \cdot 4^0 \cdot 4^0 \cdot \dots$$

$$= 4^3 \cdot 4^{1/5} \cdot 4^{2/25} \cdot 1 \cdot 1 \cdot \dots$$

$$= 4^3 \cdot \sqrt[5]{4} \cdot \sqrt[25]{4} \cdot 1 \cdot 1 \cdot \dots$$

$$6^{5/9} = 6^{.5+.05+.005+.0005+\dots} = 6^{.5} \cdot 6^{.05} \cdot 6^{.005} \cdot 6^{.0005} \cdot \dots$$

$$= 6^{\frac{1}{2}} \cdot 6^{1/20} \cdot 6^{1/200} \cdot 6^{1/2000} \cdot \dots$$

$$= \sqrt{6} \cdot \sqrt[20]{6} \cdot \sqrt[200]{6} \cdot \sqrt[2000]{6} \cdot \dots$$

$$5^{\sqrt{2}} = 5^{1+.4+.01+.004+.002+\dots} = 5^1 \cdot 5^4 \cdot 5^{.01} \cdot 5^{.004} \cdot 5^{.0002} \cdot \dots$$

$$= 5^1 \cdot \sqrt[5]{5^2} \cdot \sqrt[100]{5} \cdot \sqrt[250]{5} \cdot \dots$$

In general:

Definition 6.1.9

$$b^r = b^{t_1} \; b^{t_2} \; b^{t_3} \dots \text{ where } r \text{ is real,}$$

$$t_n \text{ is rational and } \lim_{n\to\infty} t_n = r$$

Later in this chapter we will further support the existence of real exponents when we consider exponential functions.

It is easy to believe correctly there are infinitely many irrational numbers when you consider all real exponents. It is also easy to believe correctly that any computation ignores small numbers after awhile so our display however good, must be only an approximation.

EXAMPLE 6.1.10

Approximate each of the following by using the first five terms of the infinite series expansion of the exponent. Check your results by direct calculation.

$$\sqrt[3]{8} = 8^{1/3} = 8^{.3\bar{3}} = 8^{.3} \cdot 8^{.03} \cdot 8^{.003} \cdot 8^{.0003} \cdot 8^{.00003} \cdot \ldots$$

$$\doteq 1.866\,(1.6044)(1.0063)(1.0006)(1.0001) \cdot \ldots$$

$$\doteq 2$$

$\sqrt[3]{8} = 2$ by direct calculation.

Notice that $\sqrt{8}$ is exactly 2 because $2^3 = 8$.

$5^{\sqrt{3}} \doteq 5^{1.7321}$ ($\sqrt{3} = 1.7321$ by calculator)

$= 5^{1} \cdot 5^{.7} \cdot 5^{.03} \cdot 5^{.002} \cdot 5^{.0001}$

$= 5(3.0852)(1.0495)(1.0032)(1.0002)$

$= 16.2437$

$5^{\sqrt{3}} = 16.2425$ by direct calculation.

Notice that $16.2425 < 16.2437$ because $\sqrt{3} < 1.7321$

We end this section with a summary of properties of real exponents.

Laws of Exponents 6.1.11

(a) $b^0 = 1$, if $b \neq 0$ (0^0 is not defined)

(b) $b^{-r} = \dfrac{1}{b^r}$ $b \neq 0$

(c) $b^r \cdot b^s = b^{r+s}$ $b \geq 0$

(d) $(b^r)^s = b^{rs}$ $b \geq 0$

(e) $\frac{b^r}{b^s} = b^{r-s}$ $\quad b \geq 0$

(f) $\sqrt[s]{b^r} = b^{\frac{r}{s}}$ $\quad b \geq 0$

(g) $(ab)^r = a^r b^r$ $\quad$ a and b are not both negative

(h) $(\frac{a}{b})^r = \frac{a^r}{b^r}$ $\quad$ a and b are not both negative, $b \neq 0$

Exercise Set 6.1

(1 - 6) Evaluate each of the following expressions

(a) on your calculator, as they are written

(b) on your calculator after each has been simplified

Example: $\frac{4^2 \cdot 4^{\sqrt{2}}}{4^{\frac{3}{2}}}$

(a) $\frac{4^2 \cdot 4^{\sqrt{2}}}{4^{\frac{3}{2}}} \doteq \frac{16(7.1030)}{8} \doteq 14.2060$

(b) $\frac{4^2 \cdot 4^{\sqrt{2}}}{4^{\frac{3}{2}}} = 4^{2+\sqrt{2}-\frac{3}{2}} = 4^{\frac{1}{2}+\sqrt{2}} = 4^{\frac{1}{2}} \cdot 4^{\sqrt{2}} = 2 \cdot 4^{\sqrt{2}} \doteq 14.2060$

1) $\sqrt[3]{8(-2)^{12}(-1)^9}$ $\qquad$ 2) $\sqrt[4]{81(.0625)^5}$

3) $\frac{3^{.9}(.3)^2}{.09}$ $\qquad$ 4) $\frac{5^{2/3} + 5^{2/3}}{5^{2/3}}$

5) $(2^{\sqrt{2}})^{\sqrt{2}}$ 6) $(\sqrt{3}^{\sqrt{2}})^{\sqrt{2}}$

7) $\dfrac{(2.7)^{-\sqrt{3}}}{.3^2}$ 8) $(\dfrac{\sqrt{6}}{6\sqrt{2}})^{-8}$

(9 - 14) Evaluate each of the following expressions to four decimal places.

9) $\sqrt{-74}$ 10) $\sqrt{-\pi}$

11) $2i\ \sqrt{43}$ 12) $-5i\ \sqrt{\cos 10^{\circ}}$

13) $\sqrt[5]{-9}$ 14) $\sqrt[7]{-47}$

(15 - 22) Evaluate each of the following by using a series expansion of the exponent. Compare your results with direct calculation.

15) 8^2 (use $2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{16384}$)

16) $9^{3/2}$ (use $\frac{3}{2} = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{19683}$)

17) $27^{1/3}$ (use $\frac{1}{3} = .3 + .03 + .003 + .0003 + .00003$)

18) $5^{4/7}$ (use $\frac{4}{7} = .5 + .07 + .001 + .0004 + .00002$)

19) $6.1^{\sqrt{2}}$ (use $\sqrt{2} = 1.41421$)

20) $8.6^{\sqrt{5}}$ (use $\sqrt{5} = 2.23607$)

21) π^{π} (use $\pi = 3.14159$ in the exponent)

22) $\sqrt{2}^{\sqrt{3}}$ (use $\sqrt{3} = 1.73205$)

(23 - 26) Determine the exact value of each of the following:

23) $(1734826)^2$ 24) $(12345)^3$

25) $(27)^8$ 16) $(2)^{40}$

27) Fill in the blank with >, < or = .

$(\sqrt{2})^{\sqrt{2}^{\sqrt{2}}}$ ______ $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$

6.2 Relations Between Exponents and Logarithms

Logarithms, logs for short, are a particularly important kind of exponent. A log of a number is a real number that can be characterized by the following definition:

Definition 6.2.1

$$b^{\log_b N} = N .$$

Another way to express this relation is to say:

Definition 6.2.2

The equation $x = \log_b N$ is equivalent to the equation $N = b^x$.

Definitions (6.2.1) and (6.2.2) express exactly the same idea in different ways.

EXAMPLE 6.2.3

$$10^{\log_{10}100} = 10^2 = 100 \quad (6.2.1)$$

$$2 = \log_{10}100 \text{ is equivalent to } 100 = 10^2 \quad (6.2.2)$$

$$10^{\log_{10}7} = 10^{.8451} = 7 \quad (6.2.1)$$

$$.8451 = \log_{10}7 \text{ is equivalent to } 7 = 10^{.8451} \quad (6.2.2)$$

From our definitions, the familiar laws of logarithms follow immediately from the laws of exponents.

Rules of Logarithms 6.2.4

(a) $\log_b(xy) = \log_b x + \log_b y$

(b) $\log_b(\frac{x}{y}) = \log_b x - \log_b y$

(c) $\log_b(x^y) = y \log_b x$

(d) $\log_b(\sqrt[y]{x}) = \frac{1}{y} \log_b x$

Before computing machines became prevalent the power of logs lay in the fact that by them multiplication and division were reduced to the simpler operations of addition and subtraction. Finding powers was made much easier and finding roots became reasonable. Think of how difficult it would be to find 7^{π} or $\sqrt[5]{2}$ without a calculating machine or logarithms. An "old-fashioned" calculating device, the slide rule, is designed around the theory of logarithms. A curious tidbit of math history is the fact that logs were discovered early in the 17th century before exponents were in use.

Common logarithms are logs whose base is 10. When no base is designated, as log 100 = 2, it is assumed in this text that the base is 10. Tables of logs are written in base 10 by tradition and the fact that our number system has developed around the number 10.

Logarithms can be produced for any base that is a positive number other than 1. Let's develop a four place table of logarithms for base 3.

n	1	2	3	4	5	6	7	8	9	10
$\log_3 n$	0	.6310	1	1.2620	1.4650	1.6310	1.7712	1.8930	2	2.0960

$3^0 = 1$ so $\log_3 1 = 0$

$3^1 = 3$ so $\log_3 3 = 1$

We can determine $\log_3 2$ by trial and error

$3^{.5} = 1.7321$ too small $\left(.5 = \frac{0+1}{2}\right)$ *

$3^{.75} = 2.2795$ too big $\left(.75 = \frac{.5+1}{2}\right)$

$3^{.625} = 1.9870$ too small $\left(.625 = \frac{.75+.5}{2}\right)$

$3^{.6875} = 2.1282$ too big $\left(.6875 = \frac{.625+.75}{2}\right)$

$3^{.6563} = 2.0565$ too big $\left(.6563 = \frac{.6875+.625}{2}\right)$

$3^{.6407} = 2.0215$ too big $\left(.6407 = \frac{.6563+.625}{2}\right)$

$3^{.6329} = 2.0043$ too big $\left(.6329 = \frac{.6407+.625}{2}\right)$

$3^{.6290} = 1.9958$ too small $\left(.6290 = \frac{.6329+.625}{2}\right)$

$3^{.6310} = 2.0000$ Hurrah! $\left(.6310 = \frac{.6290+.6329}{2}\right)$

The following verbal algorithm does this computation. The algorithm determines $\log_b a$ when it is already known that $\log_b c < \log_b a < \log_b d$ and $\log_b c$ and $\log_b d$ are known.

---

*Notice the averaging technique employed here. It is extremely efficient and a device that can often be used in many settings to converge on an answer.

```
1.  Remember a, b
2.  $e \leftarrow \log_b c$
3.  $f \leftarrow \log_b d$
4.  $g \leftarrow \frac{1}{2}(e+f)$
5.  Compute $b^g$
6.  If $b^g = a$ print e and stop
7.  If $b^g < a$ go to 11
8.  $f \leftarrow g$
9.  $g \leftarrow \frac{1}{2}(g+e)$
10. go to 5
11. $e \leftarrow g$
12. $g \leftarrow \frac{1}{2}(g+f)$
13. go to 5
```

The beginning of a trace for determining $\log_3 2$ is:

a	b	c	d	e	f	g	b^g
2	3	1	3	0	1	.5	1.732
				.5	.75	.75	2.2795
				.625		.625	1.9870
						.6875	2.1282

$\log_3 4 = 2 \log(2)$ by 6.2.4(c)

$= 1.2620$

$\log_3 6 = \log_3 2 + \log_3 3$ by 6.2.4(a)

$= 1.6310$

$\log_3 8 = \log_3 2 + \log_3 4$ by 6.2.4(a)

$= 1.8930$*

$\log_3 9 = 2 \log_3 3$ by 6.2.4(c)

$= 2$

$\log_3 10 = \log_3 2 + \log_3 5$ by 6.2.4(a)

$= 2.0960$

Trial and error can be employed again to determine $\log_3 5$, $\log_3 7$, and $\log_3 10$.

Even our small table of logs can be used for computation, if our calculators and computers are nearby.

EXAMPLE 6.2.5

Determine 4^5

Solution: $\log_3 4^5 = 5 \log_3 4 = 6.3100$

$3^{6.3100} = 1{,}024.7096$ so $4^5 = 1{,}024.7906$

By direct calculation $4^5 = 1024$.

Clearly 4^5 must be an integer. A little good mathematical sense tells us that 4^5 must be 1,024 since powers of 4 must end in 4 or 6.

This example points out an important fact when calculating with logarithms. No matter how accurate your tables are, you can only hope for a good approximation. Calculators and computers perform many functions by means of logarithms. This is one reason why even expensive and sophisticated machines have errors.

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*Notice that $1.9689 = 3(.6563)$ since $8 = 2^3$ so $\log_3 8 = 3\log_3 2$.

EXAMPLE 6.2.6

Determine $\sqrt[7]{8}$

Solution:

$$\log_3 \sqrt[3]{8} = \frac{1}{7} \log_3 8 = .2704$$

$$3^{.2704} = 1.3459$$

By direct calculation $\sqrt[7]{8} = 1.3459$

At first glance EXAMPLES 6.2.5 and 6.2.6 seem to be silly ways to use a calculator for we can determine the values of 4^5 and $\sqrt[7]{8}$ directly. These examples are designed to exhibit not only the properties of logarithms and exponents but to nurture an appreciation for the historical significance of logarithms as a computational tool and to recognize the genius of their invention.

Exercise Set 6.2

(1 - 10) Use our table of logarithms base 3 to determine each of the following to the nearest tenth. Check your results by direct calculation.

1) $\sqrt{6}$

2) $\sqrt{8}$

3) 8×7

4) 9×6

5) $\sqrt[5]{6}$

6) $\sqrt[6]{5}$

7) $\frac{1}{4}$

8) $\frac{1}{5}$

9) $\frac{6\sqrt{5}}{8}$

10) $\frac{5\sqrt[3]{7}}{4}$

11) $\frac{5}{3\sqrt[3]{9}}$

12) $\frac{2}{5\sqrt{7}}$

(13 - 18) Complete the following table and use it to determine each of the following to the nearest tenth. Check your results by direct calculation.

n	11	12	13	14	15
$\log_3 n$					

[*]

13) $\log_3 1.2$

14) $\log_3 1.4$

15) $\log_3 .13$

16) $\log_3 .15$

17) $\log_3 .0015$

18) $\log_3 .00012$

19) Why are your answers to (15 - 18) negative numbers?

20) Why is 0 not an acceptable base for logarithms?

21) Why is 1 not an acceptable base for logarithms?

22) Why are negative numbers not acceptable bases for logarithms?

23) Is the logarithm of a negative number negative? Why or why not.

24) If $\log_b N = 1$, what relationship is true between b and N?

25) Complete the trace of $\log_3 2$.

26) Write a program for your own calculating device for $\log_b a$ that follows the verbal algorithm of this section.

27) Show that if $\{n_1, n_2, n_3, \ldots\}$ is a geometric sequence then $\{\log_b n_1, \log_b n_2, \log_b n_3, \ldots\}$ is an arithmetic sequence where $b > 0$, $b \neq 1$.

---

[*] You may wish to do exercises 25 and 26 first and use your results.

6.3 Logarithms to Different Bases

In Section 6.2 we determined $\log_b n$ by using the relationship between logs and exponents and trial and error. We now wish to consider the relationship between logarithms of numbers to different bases.

EXAMPLE 6.3.1 Find $\log_5 2$ to four decimal places.

Let $\log_5 2 = x$

then $5^x = 2$ just as before.

Solution: $\log(5^x) = \log 2$* taking the log of both sides by 6.2.4 (c)

$x \log 5 = \log 2$

$x = \frac{\log 2}{\log 5}$ Notice that this is division of logs.

$$x = \frac{.3010}{.6990} = .4307$$

and $\log_5 2 = .4307$

Also $5^{.4307} = 2$

We use base 10 only because it is convenient, only a keystroke away. Any other base could also be used.

---

*Recall that if no base is specified it is assumed to be base 10. Thus $\log_2$ means $\log_{10} 2$.

EXAMPLE 6.3.2 Using the $\log_3 n$ chart from 6.2 find $\log_5 2$.

Solution: let $\log_5 2 = x$

$$\text{then } 5^x = 2$$

$$x \log_3 5 = \log_3 2$$

$$x = \frac{\log_3 2}{\log_3 5}$$

$$x = \frac{.6310}{1.4650} = .4307$$

and $\log_5 2 = .4307$

These two examples illustrate the following theorem:

Theorem 6.3.3

$$\log_a y = \frac{\log_b y}{\log_b a}$$

Proof: let $\log_a y = x$

$$\text{then } a^x = y$$

$$\log_b a^x = \log_b y$$

$$x \log_b a = \log_b y.$$

$$x = \frac{\log_b y}{\log_b a}$$

Notice that this proof follows exactly the same format as EXAMPLES 6.3.1 and 6.3.2.

Logarithms can be used to solve problems that appear, to the casual observer, to have no relation to logarithms.

EXAMPLE 6.3.4 How many digits are there in 2^{49}?

Solution:

let $2^{49} = 10^x$

$\log 2^{49} = \log 10^x$

$49 \log 2 = x \log 10 = x$

$14.7505 = x$

so $2^{49} \doteq 10^{14.7505}$

$10^{14} < 10^{14.7505} < 10^{15}$

10^{14} has 15 digits

10^{15} has 16 digits

so $10^{14.7505}$ has 15 digits

and 2^{49} has 15 digits.

By direct calculation $2^{49} = 5.6295 \times 10^{14}$ which has 15 digits.

Notice that for this example it is essential to use base 10.

Exercise Set 6.3

(1 - 10) Determine each of the following to four decimal places. Check your answer by using an exponential expression.

1) $\log_2 54$

2) $\log_3 150$

3) $\log_{11} 0.009$

4) $\log_4 0.416$

5) $\log_{\sqrt{2}} 6$

6) $\log_{\sqrt{3}} 7$

7) $\log_{\pi} 8$

8) $\log_{\pi} 1$

9) $\log_{.03}10$ 10) $\log_{.7}9$

(11 - 14) Determine each of the following:

11) $\log_{\sqrt{2}}\sqrt{2}$ 12) $\log_{\pi}\pi$

13) $\log_a\sqrt{a}$ 14) $\log_x x^2$

(15 - 20) Determine how many digits are in each of the following numbers.

15) 4^{20} 16) 5^{18}

17) 127^{19} 18) 253^{12}

19) 57^{90} 20) 63^{85}

(21 - 26) Determine the value of each of the following expressions:

21) $\dfrac{\log_3 81 + \log_2 64}{\log_{10} 10,000}$ 22) $\dfrac{\log_2 16 + \log_2 1}{\log_5 25 + \log_2 64}$

23) $\dfrac{2 + \log_2(\log_5 5)}{5\log_{0.36}\frac{3}{5}}$ 24) $\dfrac{\log_{1.2}\frac{25}{36} \div \log_2 4\sqrt{2}}{\log_4\sqrt{2} - \log_3 9\sqrt{3}}$

25) $5 + 0^{\log\sqrt{2}} - 2(\log\sqrt{2})^0 + \log 10 + \log_{\sqrt{2}}1$

26) $6 + 7^{\log_7 2} - 3\log .001 - \log_7 7$

(27 - 30) We wish to determine a relationship between $\log_a b$ and $\log_b a$.

27) Complete the following charts

$\log_2 3$	$\log_3 2$

$\log_5 7$	$\log_7 5$

$\log_{10} 2$	$\log_2 10$

28) Create 5 more $\log_a b$ and $\log_b a$ charts of your own.

29) Write a relationship between $\log_a b$ and $\log_b a$. (Hint: (multiply them together)

30) Prove your relationship in (29). You may want to use $\log_a b = x$ and $\log_b a = y$ and convert these log expressions to exponential expressions.

31) Write a verbal algorithm to compute $\log_b a$ assuming that your computing device has a log key or log command.

32) Write a program for $\log_b a$ using your answer to exercise 31.

6.4 Natural Logarithms and e

In this section we will again study an illusive number that is the base for a special kind of logarithm called natural logarithms.

EXAMPLE 6.4.1 Consider the following sequence:

$$(1 + \frac{1}{1})^1,\ (1 + \frac{1}{2})^2,\ (1 + \frac{1}{3})^3,\ (1 + \frac{1}{4})^4, \ldots,\ (1 + \frac{1}{k})^k,\ \ldots$$

To determine the first 10 terms of this sequence we can use the following verbal algorithm:

1. $N \leftarrow 1$, display N (N is counter, M is term)
2. Determine $M = (1 + \frac{1}{N})^N$, display M.
3. $N \leftarrow N + 1$
4. When N = 11 stop, otherwise go to step 2.

The following are programs for this algorithm.

HP-33E

```
 1   0                 8   1               15   RCL 1
 2   STO 1             9   +               16   f x ≠ y
 3   1                10   RCL 1           17   GTO 03
 4   STO + 1          11   f y^x           18   RCL 2
 5   RCL 1            12   R/S (term)      RUN, RTN, store 10 in R2,
 6   R/S (counter)    13   RCL 2           R/S
 7   g  1/x           14   ENTER
```

TI 58

```
00   0                11   1/x             22   x >< t
01   STO              12   +               23   RCL
02   01               13   1               24    01
03   2nd Lbl          14   =               25   INV 2nd
04   A                15   y^x             26   x = t
05   1                16   RCL             27    A
06   SUM              17    01             28   R/S (stops program)
07   01               18   =               29   RST
08   RCL              19   R/S (term)      LRN, RST, store 10 in R2,
09   01               20   RCL             R/S
10   R/S (counter)    21    02
```

TRS-80

```
10  N = 1                    50   If N = 11  GO TO 70
20  M = (1 + 1/N)↑N          60   GO TO 20
30  PRINT N,M                70   END
40  N = N + 1
```

Load the program into your calculating device and fill in the following table. Find a_n to 4 decimal places.

n	a_n
1	
2	
3	2.3704
4	
5	

n	a_n
6	
7	
8	
9	
10	2.5937

Finding the 100th term of this sequence would take a long time (more than 30 minutes) by this method. But we can find the 100th term directly* by calculating $(1 + \frac{1}{100})^{100} = 2.7048$

Verify each of the following to 8 decimal places

10,000th term = 2.71826823

100,000th term = 2.71828047

10,000,000th term = 2.71828169

100,000,000th term = 2.71828182

500,000,000th term = 2.71828182

It seems safe to assume that the sequence

$$1,\ (1 + \tfrac{1}{2})^2,\ \ (1 + \tfrac{1}{3})^3,\ \ (1 + \tfrac{1}{4})^4,\ \ldots,\ (1 + \tfrac{1}{k})^k,\ \ldots$$

---

*Notice that this sequence is neither arithmetic or geometric.

approaches 2.7182818^+ when k increases without bound.[*] The limit of this infinite sequence can be expressed as:

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.7182818^+$$

EXAMPLE 6.4.2 Consider the following series:

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{k!} + \dots$$

The sequence of partial sum of this series begins:

$$\$_1 = 1 \text{ [**]}$$
$$\$_2 = 1 + 1 = 2$$
$$\$_3 = 1 + 1 + \tfrac{1}{2} = 2.5$$
$$\vdots$$

To determine the terms in the sequence of partial sums of this series we can use the following verbal algorithm:

```
1. A ← 1, B ← 1, C ← 1, D ← 1   (A is counter, D is term)
2. Display A and D.
3. A ← A + 1, D ← D + 1
4. Display A and D.
5. B ← B + 1
6. Compute C ← C * B
7. Compute 1/C
8. Compute D ← D + 1/C
9. A ← A + 1
```

---

[*] 2.7182818^+ means a number a little bigger than 2.7182818.

[**] Recall that, by definition, 0! = 1.

10. Display A and D.

11. Go back to step 5.

12. When you have reached the sum you want, stop.

The following are programs for this algorithm:

HP 33E (R$_0$ is counter, R$_3$ is term)

```
01   1                 13   STO + 3           25   STO + 1
02   STO 0             14   RCL 3             26   RCL 1
03   STO 1             15   R/S (term)        27   STO X 2
04   STO 2             16   GSB 24            28   RCL 2
05   STO 3             17   1                 29   g 1/x
06   R/S (counter)     18   STO + 0           30   STO + 3
07   R/S (term)        19   RCL 0             31   g RTN
08   1                 20   R/S (counter)     RUN, gRTN, f fix 9,
09   STO + 0           21   RCL 3             R/S
10   RCL 0             22   R/S (term)
11   R/S (counter)     23   GTO 16
12   1                 24   1
```

TI 58 (R$_{00}$ is counter, R$_{03}$ is term)

```
00   1                 21   03                42   01
01   STO               22   R/S (term)        43   RCL
02   00                23   2nd Lbl           44   01
03   STO               24   B                 45   2nd PRD
04   01                25   SBR               46   02
05   STO               26   A                 47   RCL
06   02                27   1                 48   02
07   STO               28   SUM               49   1/x
08   03                29   00                50   SUM
09   R/S (counter)     30   RCL               51   03
10   R/S (term)        31   00                52   INV SBR
11   1                 32   R/S (counter)     LRN, RST, R/S
12   SUM               33   RCL
13   00                34   03
14   RCL               35   R/S (term)
15   00                36   GTO
16   R/S (counter)     37   B
17   1                 38   2nd Lbl
18   SUM               39   A
19   03                40   1
20   RCL               41   SUM
```

TRS-80 (for 1st 16 terms; A is the counter, D is the term)

```
  10    DEFBL D
  20    A = 1:B = 1: C = 1: D = 1
  30    PRINT A, D
  40    A = A + 1
  50    D = D + 1
  60    PRINT A, D
  70    B = B + 1
  80    C = C * B
  90    D = D + 1/C
 100    A = A + 1
 110    PRINT A, D
 120    IF A = 16 GO TO 130
 130    GO TO 60
 140    END
```

Load the program into your calculating device and fill in the following table.

Find S_n to 8 decimal places.

n	S_n	n	S_n
1		9	
2		10	
3		11	
4		12	2.71828183
5		13	
6	2.71666667	14	2.71828183
7		15	
8	2.71825396	16	

It seems safe to assume that the series

$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{k!} + \ldots$ approaches 2.7182818^+

when k increases without bound. This infinite sum can be expressed as:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 2.7182818^+$$

This number 2,7182818$^+$ looks like a very special number. Indeed it is! It is the number e. One of the many ways to define e is the following:

Definition 6.4.3

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$e \doteq 2.7182818^+$$

e is an irrational number so it has no exact representation as a decimal. e is also a transcendental number. This means that e cannot be expressed as the root of a polynomial equation.* e is probably called "e" because it represents a special kind of exponent (as you will see later) and because it was written as "e" when it was first developed by the great mathematicians Euler and Gauss. Another of Euler's important discoveries is the remarkable equation involving five important constants:

$$e^{\pi i} + 1 = 0$$

e is the base for natural logarithms which are usually written as ℓn rather than $\log_e$.

Definition 6.4.4

$$x = \ell n\ y \text{ is equivalent to } y = e^x$$

---

*Other transcendental numbers are π, log 2, $2^{\sqrt{2}}$. If this idea intrigues you refer to Numbers: Rational and Irrational by Ivan Niven, Volume 1, New Mathematical Library, the Mathematical Association of America, 1961.

Natural logarithms are the "best" logarithms to use in many applications of the calculus and in modern analysis. You will see a lot more of the important and surprising constant, e, in your future mathematical career.

Exercise Set 6.4

1) Determine how to display e on your calculating device.

(2 - 7) Refer back to EXAMPLE 6.4.1.

2) Find the 500th term of the sequence.

3) Find the 5000th term of the sequence.

4) Find the 10^9 term of the sequence.

5) Find the 10^{10} term of the sequence.

6) Why is your answer to (5) "strange"?

7) Write a few sentences to convince another student that $\lim_{n \to \infty} (1 + \frac{1}{n})^n = e \neq 2.718281828$.

(8 - 11) Load your calculating device with the program of EXAMPLE 6.4.1. How long does it take your calculating device to determine:

8) the 20th term of the sequence

9) the 50th term of the sequence

Estimate how long it would take your calculating device to determine:

10) the 10^9 term of the sequence

11) the 10^{10} term of the sequence

(12 - 15) Load your calculating device with the program of EXAMPLE 6.4.2. Find each of the following. For calculators: Clear the registers before each run. Store the appropriate number in R_2. For computers: Alter line 50.

12) $\sum_{n=0}^{20} \frac{1}{n!}$ 13) $\sum_{n=0}^{30} \frac{1}{n!}$

14) Is $\sum_{n=0}^{20} \frac{1}{n!} = \sum_{n=0}^{30} \frac{1}{n!}$? Explain.

15) Write a few sentences to convince another student that

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e \neq 2.718281828.$$

16) Determine a relationship between $\ln 10$ and $\log_{10} e$.

17) Prove the relationship you developed in exercise 16.

(18 - 21) Decide whether each of the following is true or false. Give a reason to support your answer.

18) $\ln x < \log x$ for $x > 0$

19) $\ln \left(\frac{x}{y}\right) = \ln x - \ln y$

20) $\ln x^p = \ln px$

21) $e^{\ln e} = 1$

(22 - 26) Continued fractions* were studied by the great mathematicians of the 17th and 18th century and are the focus of mathematical research even today. A continued fraction is a fraction that continues.

---

*A good reference is Continued Fractions by D. C. Olds, Vol. 9, New Mathematical Library, the Mathematical Association of America, 1963.

Example: $$\frac{1}{2} = \cfrac{1}{1+\cfrac{1}{1}}$$

$$\frac{29}{67} = 0 + \cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\frac{1}{2}}}}$$

Continued fractions can be infinite.

Example: $$\sqrt{2} = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}}$$

Are you ready? $$e = 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{5+\dots}}}}}$$

22) Verify the continued fraction representation for $\frac{29}{67}$.

(23 - 24) Approximate $\sqrt{2}$ by

23) $1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+1}}}$

24) $1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+1}}}}}$

(25 - 27) Approximate e by

25) $2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+4}}}}$

26) $2+\cfrac{1}{1+\cfrac{1}{\ddots+\cfrac{6}{7+7}}}$

27) What is wrong with the following "proof"?

$$e^{i\pi}+1=0$$

$$e^{i\pi}=-1$$

$$\sqrt{e^{i\pi}}=\sqrt{-1}$$

$$e^{\frac{i\pi}{2}}=i$$

$$e^{2i\pi}=1=e^{0}$$

$$2i\pi=0$$

$$2\neq 0 \text{ so } i=0 \text{ or } \pi=0$$

6.5 Natural Growth and Decay, and Applications

The name "natural" is appropriate for natural logarithms because of their appearance in various laws of growth and decay.

Radioactive materials decay at a rate proportional to the amount of material present. This is characterized by the equation:

$$y = c \cdot e^{kt}$$

y is the amount of material remaining at time t

c is the amount of material present at time zero

k depends on the particular substance and the units being used.

EXAMPLE 6.5.1 (Radioactive Decay)

A physicist has 20 grams of radioactive radium. The half-life of radium is 1600 years.* How much of the radium will be present after (a) 2000 years

(b) t years

(c) How long will it take for 2 of the 20 grams of radium to disappear?

---

*A given amount of radium will be half-gone after 1600 years.

(a) We must first determine k:

$$10 = 20e^{k(1600)}$$

$$\ln 10 = \ln 20e^{1600k}$$

$$\ln 10 = \ln 20 + 1600k \ln e$$

$$\ln 10 = \ln 20 + 1600k$$

$$\frac{\ln 10 - \ln 20}{1600} = k$$

$$-.0004 \doteq k$$

$$y \doteq 20e^{-.0004(2000)}$$

$$y \doteq 20e^{-.8}$$

$$y \doteq 8.9866$$

There will be approximately 8.9866 grams left after 2000 years.

(b) $y \doteq 20e^{-.0004t}$

(c) When 2 grams disappear there will be 18 grams left.

$$18 \doteq 20e^{-.0004t}$$

$$\ln 18 \doteq \ln (20e^{-.0004t})$$

$$\ln 18 \doteq \ln 20 + (-.0004t)\ln e$$

$$\ln 18 \doteq \ln 20 - .0004t$$

$$\frac{\ln 18 - \ln 20}{-.0004} \doteq t$$

$$263.4013 \doteq t$$

Two grams will disappear in approximately 263 years.

Many large populations, for example people and bacteria, under ideal conditions grow at a rate proportional to the size of the population present. This is also characterized by the equation:

$$y = c \cdot e^{kt}.$$

y is the size of the population at time t

c is the size of the population at time zero (your reference point)

k depends on the particular population being considered and the units being used.

EXAMPLE 6.5.2 (Population Growth)

The number of bacteria at 9 A.M. is estimated to be 1000. At noon the bacteria count is estimated to be 8500. (a) How many bacteria are there after t hours?

(b) When will there be approximately 20,000 bacteria?

(a) We first need to find k.

$$\begin{aligned} 8500 &= 1000e^{k(3)} \\ \ln 8500 &= \ln 1000 + 3k \ln e \\ \ln 8500 &= \ln 1000 + 3k \\ \frac{\ln 8500 - \ln 1000}{3} &= k \\ .7134 &\doteq k \end{aligned}$$

$$y = 1000e^{.7134t}$$

There will be approximately $1000e^{.7134t}$ bacteria after t hours.

(b)
$$\begin{aligned} 20{,}000 &\doteq 1000e^{.7134t} \\ 20 &\doteq e^{.7134t} \\ \ln 20 &\doteq .7134t \ln e \\ \frac{\ln 20}{.7134} &\doteq t \\ 4.1992 &\doteq t \\ \text{4 hrs. 12 min.} &\doteq t \end{aligned}$$

At approximately 1:12 PM there will be 20,000 bacteria.

EXAMPLE 6.5.3 (Bank Interest)

As seen in Chapter 2, if a bank gives interest at a rate compounded annually, the amount in an account is given by the formula

$$A = P\ (1 + i)^n.$$

A is the total in the account after n years
P is the initial deposit,
i is the rate of interest expressed as a decimal.

(a) Find the total amount of $1000 invested at a rate of 6% for 10 years. $A = \$1000.\ (1 + .06)^{10}$

$$= \$1790.85$$

If the bank gave compound interest the formula would be

$$A = P\ (1 + \frac{i}{q})^{nq}$$

where q is the number of times the interest is compounded per year.

The expression $(1 + \frac{i}{q})^{nq}$ should look familiar. i is fixed and small. q and nq are getting very large. Recall that $\lim_{n \to \infty} (1 + \frac{1}{n})^n = e$. So, if banks compounded their interest without restriction (continuously) the maximum total amount would be determined by

$$A = P \cdot e^{rt}$$

r is the yearly rate in decimal form

A is the total accumulation after t years

P is the principal (amount invested).

Continuously compounded, interest is a "natural" growth.

Relation 6.5.4

Natural growth and decay at a rate proportional to the number present is characterized by the relation:

$$y = c \cdot e^{kt}$$ [*]

EXAMPLE 6.5.5 (Rocket Launching[*])

The velocity gained by a launch vehicle when its propellant is burned to depletion is expressed by the equation $v = c \cdot \ln \cdot R$. The launch velocity gained during the burn is v, the exhaust velocity is c and R is the mass ratio. The mass ratio R is defined by

---

[*] Refer to page 6.5 - 1 for explanation of the variables.

[**] This information is provided courtesy of the National Aeronautics and Space Administration publication Space Mathematics pp. 72-74.

$R = \frac{\text{takeoff weight}}{\text{burnout weight}}$. This ratio applies to either the entire launch vehicle or just a single stage. The takeoff weight consists of propellant (fuel), F, structure, S, and payload, P. Thus the mass ratio is

$$R = \frac{F + S + P}{S + P} .$$

At burnout all the fuel has been consumed so $F = 0$. It has been found that the weight of fuel cannot be more than about 10 times the weight of the structure.

(a) What is the largest possible value for R?

$$R = \frac{10S + S + P}{S + P} = \frac{11S + P}{S + P} = \frac{11S + 11P - 10P}{S + P}$$

$$= 11 - \frac{10P}{S + P} \leq 11$$

If $R = 11$ then $P = 0$ so the rocket would have no room for a payload.

(b) If the highest energy propellant available for takeoff from Earth has an average exhaust velocity of 9,600 feet per second, determine the velocity of a launch vehicle with $R = 10$

$$v = 9{,}600 \ln 10$$

$$= 22{,}104.82 \text{ feet per second.}$$

The total launch velocity of a vehicle must be about 30,000 feet per second so this vehicle cannot fly. Even if R = 11 the launch velocity would not be 30,000 feet per second.

(c) What must the exhaust velocity be so that the vehicle in (b) could be placed into orbit?

$$30{,}000 = c \ln 10$$

$$13{,}028.8345 \text{ ft/sec} = c$$

The solution to this problem is staging. As soon as the propellant in the first stage has been burned, the stage is jettisoned and permitted to fall into the ocean so that the vehicle is freed of deadweight. The second and subsequent stages become more effective and orbit is possible.

Exercise Set 6.5

1) Assume that the number of bacteria B present after t minutes is given by the formula $N = 200\, e^{.27t}$.

   a) How many bacteria are present after ½ hour?

   b) How many bacteria are present after 2 hours?

   c) How long will it take to have 15,000 bacteria?

2) The half-life of radioactive bismuth is 5 days. If a scientist has 15 grams,

   (a) How much of the bismuth will disappear after 12 days?

   (b) How long will it take for 6 grams to disappear?

3) One gram of radioactive lead is made. At the end of two hours .04 gram remains. What is the half-life of radioactive lead?

4) In a certain culture bacteria grow at a rate proportional to the number present. At a certain instant the number of bacteria present is 1000. The number present 10 hours later is 7000. How long does it take the population to double?

5) Smart Sidney deposits $5000 in a bank that continuously compounds its 5% annual interest. How much does Sidney have after 7 years?

6) Sidney deposited $5000 as described in exercise 5. Susie deposited $5000 at a bank that compounded its 5.5% interest monthly. If they both intend to leave their money in their respective banks for 5 years, who made the better investment?

7) The population of Buffalo, New York was 532,759 in 1960 and 462,768 in 1970. Assuming natural "growth", predict the population of Buffalo in 1980.

8) In 1960 the population of Atlanta, Georgia was 487,000. In 1970 the population was 497,000. Assuming natural growth, when will the population be 1,000,000?

9) The world's population doubles in about 33 years. In 1970 the world population was about 3,000,000,000. Assuming normal growth, what is a reasonable projection for the world's population in 2000?

10) In solution, sugar decomposes at a rate proportional to the amount still undissolved.

(a) Express this relation as an exponential equation.

(b) If 30 lbs. of sugar reduces to 11 lbs. in four hours, when will 80% of the sugar be decomposed?

11) Many satellites have radioisotope power supplies. The power output in watts of such a supply is given by the equation $P = 50\ e^{-t/250}$ where t is the time in days.

(a) How much power will be available at the end of 1 year?

(b) What is the half-life of the power supply?

(c) If the equipment aboard the satellite requires 10 watts of power to operate properly, what is the operational life of the satellite?

12) An approximate rule for atmospheric pressure at altitudes less than 50 miles is that standard atmospheric pressure, 14.7 pounds per square inch, is halved for each 3.25 miles of vertical ascent.

(a) Write an exponential equation to express this rule.

(b) Determine the atmospheric pressure at 20 miles.

(c) Find the altitude at which the pressure is 25% of standard atmospheric pressure.

(d) What altitude is just above 99 percent of the atmosphere? (Pressure and density are proportional; the desired altitude is the point at which the pressure is 1% of standard atmospheric pressure.)

6.6 Exponential and Logarithmic Equations

Our knowledge of exponents and logs allows us to solve many very sophisticated (and complicated-looking) equations.

An equation in which the unknown appears in an exponent is called an exponential equation. An equation in which there is the logarithm of an expression involving an unknown is called a logarithmic equation.

EXAMPLE 6.6.1 Solve for x and check:

$$5 \cdot 3^{2x+1} = 11^{1-x}$$

$$\log (5 \cdot 3^{2x+1}) = \log (11^{1-x})$$

$$\log 5 + (2x+1) \log 3 = (1-x) \log 11$$

$$\log 5 + 2x \log 3 + \log 3 = \log 11 - x \log 11$$

$$2x \log 3 + x \log 11 = \log 11 - \log 5 - \log 3$$

$$x (2 \log 3 + \log 11) = \log 11 - \log 5 - \log 3$$

$$x = \frac{\log 11 - \log 5 - \log 3}{2 \log 3 + \log 11}$$

$$x \doteq -.0675 \text{ (to 4 decimal places)}$$

Check:

$5 \cdot 3^{2(-.0675) + 1}$	$11^{1 - (-.0675)}$
$5 \cdot 3^{.8650}$	$11^{1.0675}$
12.9324	12.9326

This problem could also be easily done using natural logs. Logs to any base could be used but log and $\ln$ are easily accessible on calculating machines.

EXAMPLE 6.6.2 Solve for x and check:

$$x^{\log x} = 10{,}000\, x^3$$

$$\log (x^{\log x}) = \log (10{,}000\, x^3)$$

$$\log x \cdot \log x = \log 10{,}000 + 3 \log x$$

$$(\log x)^2 = 4 + 3 \log x$$

$$(\log x)^2 - 3 \log x - 4 = 0$$

$$(\log x-4)(\log x+1) = 0$$

$$\log x-4 = 0 \quad \text{or} \quad \log x+1 = 0$$

$$\log x = 4 \qquad \log x = -1$$

$$x = 10{,}000 \qquad x = .1$$

Check:

$$\begin{array}{l|l} 10{,}000^{\log 10{,}000} & 10{,}000\, (10{,}000^3) \\ 10{,}000^4 & 10{,}000^4 \end{array}$$

$$\begin{array}{l|l} .1^{\log .1} & 10{,}000\, (.1)^3 \\ .1^{-1} & 10{,}000\, (.001) \\ 10 & 10 \end{array}$$

Clearly log is the only reasonable choice here.

EXAMPLE 6.6.3 Solve for x and check:

$$e^{2x} - 2e^x + 1 = 0$$

$$(e^x-1)(e^x-1) = 0$$

$$e^x - 1 = 0$$

$$e^x = 1$$

$$\ln e^x = \ln 1$$

$$x = 0$$

Check:

$$\begin{array}{r|l} e^{2(0)} - 2e^{0} + 1 & 0 \\ e^{0} - 2(1)+1 & \\ 1 - 2 + 1 & \\ 0 & \end{array}$$

EXAMPLE 6.6.4 Solve for x and check

$$\ln \ln (x + 1) = .7$$

$$e^{\ln \ln (x+1)} = e^{.7}$$

Since $e^{\ln N} = N$, $\ln (x+1) = e^{.7}$

$$e^{\ln x+1} = e^{e^{.7}}$$

$$x+1 = e^{e^{.7}}$$

$$x+1 = 7.4914$$

$$x = 6.4914$$

Check:

$$\begin{array}{r|l} \ln \ln (6.4914 + 1) & .7 \\ \ln \ln (7.4194) & \\ \ln 2.0138 & \\ .7 & \end{array}$$

EXAMPLE 6.6.5

$$\begin{aligned} \log_5 \sqrt{3x+4} &= 1.73 \\ 5^{\log_5 \sqrt{3x+4}} &= 5^{1.73} \\ \sqrt{3x+4} &\doteq 16.1889 \\ 3x+4 &= 262.0814 \\ x &= 86.0271 \end{aligned}$$

Check:

$$\begin{array}{c|c} \log_5 \sqrt{3(86.0271)+4} & 1.73 \\ \log_5 16.1889 & \\ \dfrac{\ln 16.1889}{\ln 5} & \\ 1.73 & \end{array}$$

EXAMPLE 6.6.6

$$x = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}$$

A first attempt at determining a more "ordinary" form for x might involve evaluating the finite expression

$$x \doteq \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}}$$

$$x \doteq 1.9998$$

Since $x > 1.9998$ it seems reasonable to expect $x = 2$. But, this is clearly not a good enough procedure.

$$x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}$$

So $$x = \sqrt{2 + x}$$

$$x^2 = 2 + x$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

$$x - 2 = 0 \qquad x + 1 = 0$$

$$x = 2 \qquad x = -1$$

$$x > 0 \text{ so } x \neq -1$$

so indeed $x = 2$.

This example is not related to logarithms and only remotely related to exponents, but it is an interesting kind of mathematical expression. The technique is worth remembering, especially if you want someday to be a star in a math contest.

Exercise Set 6.6

(1 - 4) Solve for n.

1) $\ell = ar^{n-1}$ 2) $x = ab^{cn}$

3) $x = \log_3 n$ 4) $x^2 = 3 + \ell n\, n$

(5 - 18) Solve for x to four decimal places. Check your answers.

5) $5^{7x} = 7^{5x+1}$ 6) $3^{x+2} = 5^{x-2}$

7) $\log \log (x+1) = -.2$ 8) $\ell n\, \ell n\, (x-3) = -1.2$

9) $\log_5 (x+1) + \log_5 (x+2) = 1$

10) $\log_3 (x+3) - \log_3 x = 2$

11) $\dfrac{\ln (7x-12)}{\ln x} = 2$ 12) $\dfrac{\log (5x-6)}{\log x} = 2$

13) $x^{\ln x} = 1.5$ 14) $x^{\log x} = 2$

15) $e^{-2x} - 3e^{-x} + 2 = 0$ 16) $e^{2x} - e^{x} = 6$

17) $\ln (\ln x) = 1.2$ 18) $\log (\log x) = -.3$

(19 - 20) Solve for x:

19) $x = \sqrt{3 + \sqrt{3 + \sqrt{3 + \ldots}}}$ 20) $x = \sqrt{5 + \sqrt{5 + \sqrt{5 + \ldots}}}$

(21 - 24) A technique similar to that employed in EXAMPLE 6.6.6 can be used to evaluate infinite continued fractions.*

EXAMPLE: $x = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}$

$$x - 1 = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}$$

$$x = 1 + \frac{1}{2 + x-1}$$

$$x = 1 + \frac{1}{x+1}$$

$$x(x+1) = x+1 + 1$$

$$x^2 + x = x + 2$$

$$x^2 = 2$$

$$x = \sqrt{2}$$

---

*Refer to 6.4 exercises 21 - 25.

Solve for x:

21) $$x = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \dots}}}}$$

22) $$x = 3 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \cfrac{1}{6 + \dots}}}}}$$

6.7 Graphs of Exponential and Logarithmic Functions

In this section we will carefully consider the graphs of exponential and logarithmic functions. These graphs convey important and characteristic information about these two classes of functions.

EXAMPLE 6.7.1

Graph the following functions:

$f(x) = 1.2^x$

$f(x) = 1.4^x$

$f(x) = 1.7^x$

$f(x) = 2^x$

$f(x) = e^x$

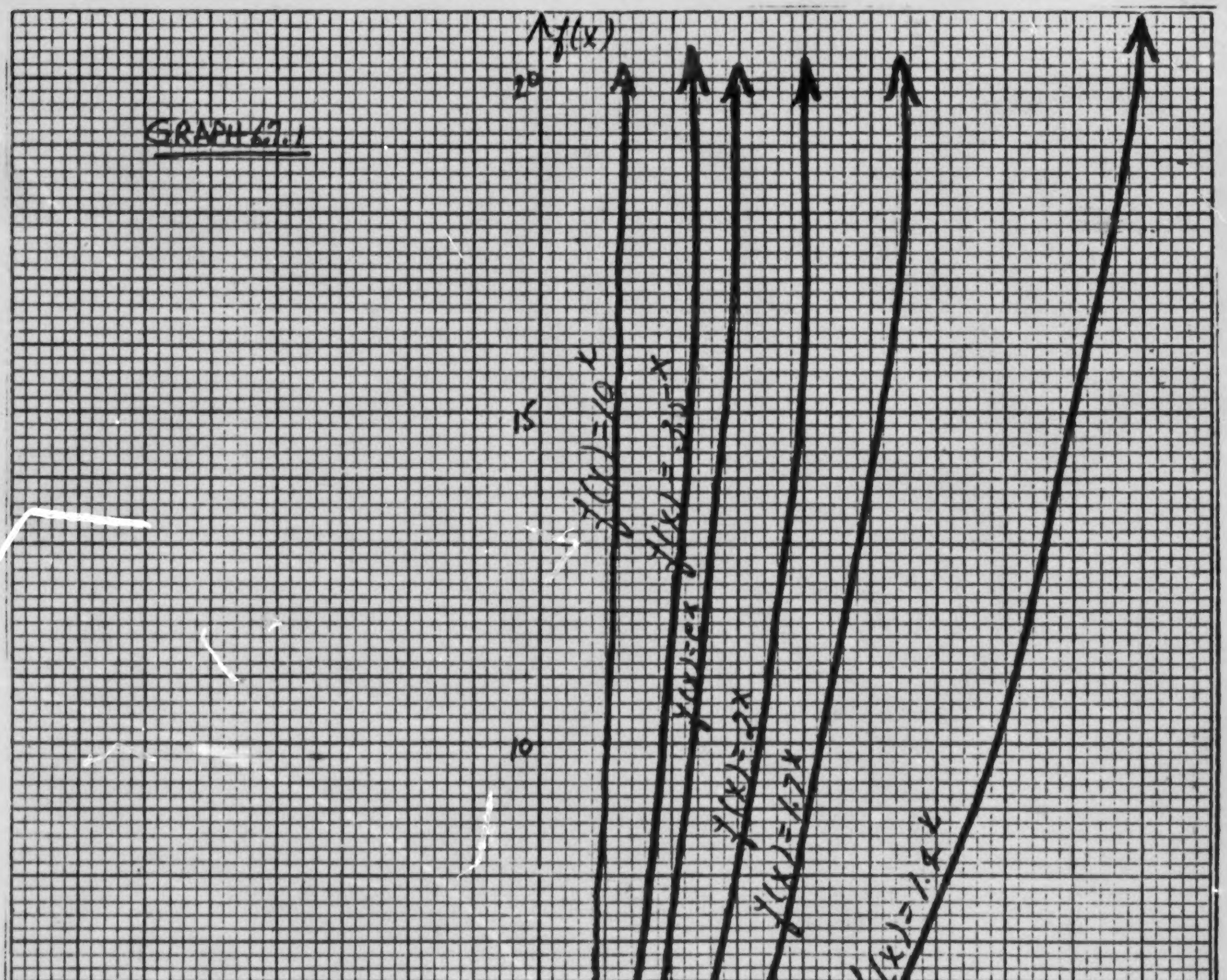
$f(x) = 3.5^x$

$f(x) = 10^x$

1) From the graphs of each of these functions determine the

    (a) domain
    (b) range
    (c) $f(0)$
    (d) $f(1)$
    (e) the character of $f(x)$ as $x \longrightarrow -\infty$
    (f) the character of $f(x)$ as $x \longrightarrow +\infty$
    (g) the continuity or discontinuity of $f(x)$
    (h) the character of $f(x)$ as x increases

2) Make a list of generalizations about $f(x) = b^x$ when $b > 1$.


GRAPH 6.7.1
f(x)
20
15
10
5
x
-7
-4
0
4
9
f(x)=10^x
f(x)=3.5^x
f(x)=e^x
f(x)=2^x
f(x)=1.7^x
f(x)=1.4^x
f(x)=1.2^x

Solution: Using the techniques of graphing developed in section 5.1 we can program our calculating device to compute ordered pairs for each of these functions. (Use $d \leq .5$). We obtain the preceding graph.

1.

$f(x)$	1.2^x	1.4^x	1.7^x	2^x	e^x	3.5^x	10^x
a	all reals	all reals	all reals	all reals	all reals	all reals	
b	$\{x \mid x > 0\}$	$\{x \mid x > 0\}$	$\{x \mid x > 0\}$	$\{x \mid x > 0\}$	$\{x \mid x > 0\}$	$\{x \mid x > 0\}$	$\{x \mid x >$
c	1	1	1	1	1	1	1
d	1.2	1.4	1.7	2	e	3.5	10
e	$f(x) \to 0$	$f(x) \to 0$	$f(x) \to 0$	$f(x) \to 0$	$f(x) \to 0$	$f(x) \to 0$	$f(x) \to 0$
f	$f(x) \to \infty$	$f(x) \to \infty$	$f(x) \to \infty$	$f(x) \to \infty$	$f(x) \to \infty$	$f(x) \to \infty$	$f(x) \to \infty$
g	continuous	continuous	continuous	continuous	continuous	continuous	continuous
h	increases	increases	increases	increases	increases	increases	increases

2. The class of functions $f(x) = b^x$ $b > 1$ can be characterized as follows:

   (a) The domain is all real numbers

   (b) The range is all positive real numbers

   (c) $f(0) = 1$. The graph passes through the point $(0,1)$

   (d) $f^{-1}(b) = 1$. The graph passes through the point $(1,b)$

   (e) $\lim\limits_{x \to -\infty} f(x) = 0$

   (f) $\lim\limits_{x \to +\infty} f(x) = +\infty$

   (g) They are continuous throughout their domain

   (h) As x increases, f(x) increases*

---

*In calculus, this characteristic is called monotonic increasing.

We have been able to compile an impressive list. Many of these statements will not be proved until you study the Calculus. Each statement is true.

EXAMPLE 6.7.2

Graph the following functions

$f(x) = \log_{1.2}x$

$f(x) = \log_{1.4}x$

$f(x) = \log_{1.7}x$

$f(x) = \log_{2}x$

$f(x) = \ln x$

$f(x) = \log_{3}x$

1) From the graphs of these functions determine the:

   a) domain

   b) range

   c) $f^{-1}(0)$

   d) $f^{-1}(1)$

   e) the character of $f(x)$ as $x \longrightarrow 0^+$ *

   f) the character of $f(x)$ as $x \longrightarrow +\infty$

   g) the continuity or discontinuity of $f(x)$

   h) the character of $f(x)$ as x increases

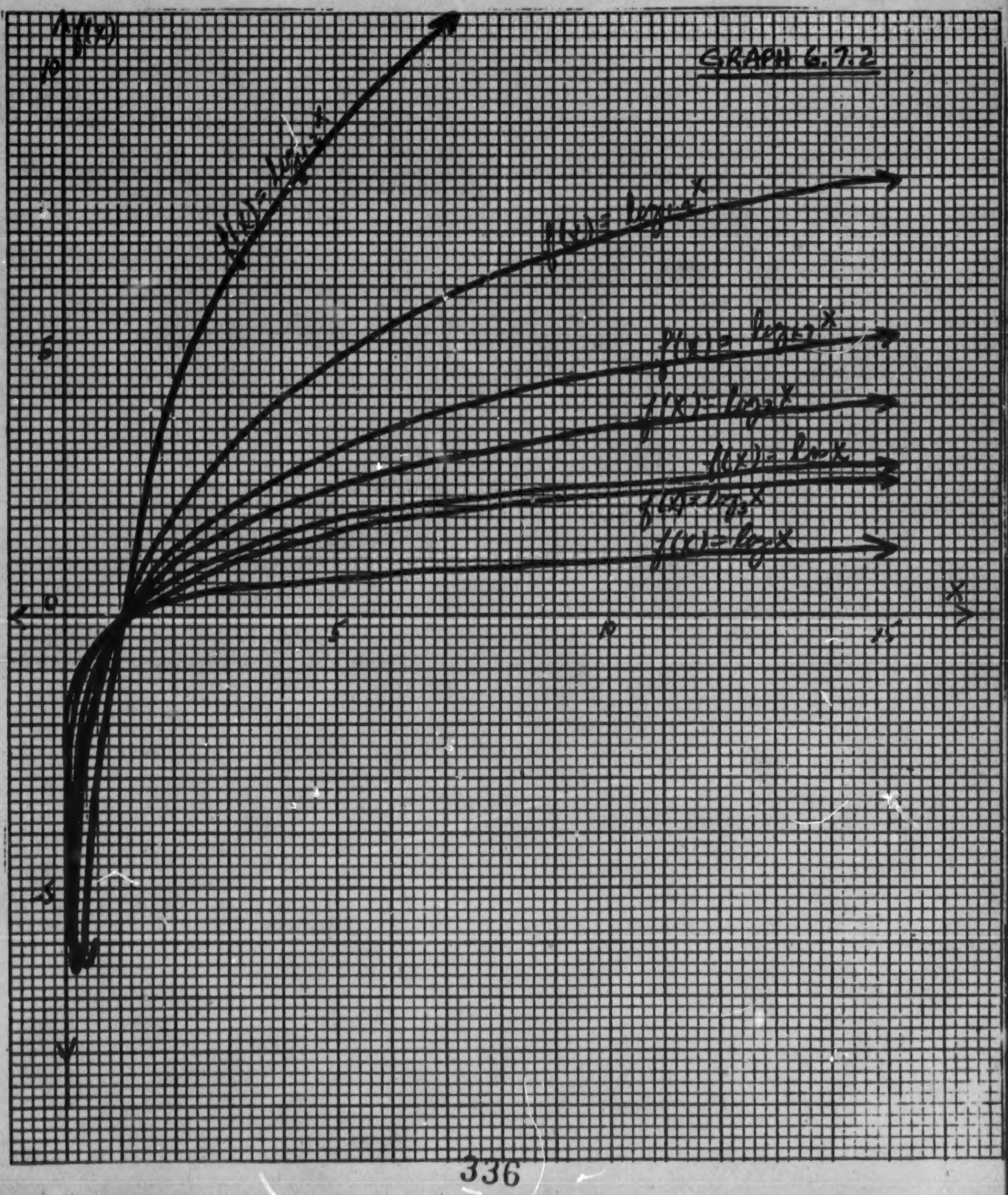
   i) the inverse of $f(x)$ (denoted $f^{-1}(x)$)

2) Make a list of generalizations about $f(x) = \log_b x$ when $b > 1$.

---

* $x \longrightarrow 0^+$ means as x approaches zero from the positive direction.

Solution: Again using the techniques of graphing developed in section 5.1, we can program our calculating device to compute ordered pairs for each of these functions. (Use $d \leq .5$ and $\log_b x = \frac{\log x}{\log b}$. Store b in R_4.) We obtain the following graph.



1.

$f(x)$	$\log_{1.2}x$	$\log_{1.4}x$	$\log_{1.7}x$	$\log_2 x$	$\ln x$	$\log_3 x$	$\log x$
a	$\{x \mid x > 0\}$	$\{x \mid x > 0\}$	$\{x \mid x > 0\}$	$\{x \mid x > 0\}$	$\{x \mid x > 0\}$	$\{x \mid x > 0\}$	$\{x \mid x > 0$
b	all reals	all reals	all reals	all reals	all reals	all reals	all reals
c	1	1	1	1	1	1	1
d	1.2	1.4	1.7	2	e	3	10
e	$f(x) \to -\infty$	$f(x) \to -\infty$	$f(x) \to -\infty$	$f(x) \to -\infty$	$f(x) \to -\infty$	$f(x) \to -\infty$	$f(x) \to -\infty$
f	$f(x) \to \infty$	$f(x) \to \infty$	$f(x) \to \infty$	$f(x) \to \infty$	$f(x) \to \infty$	$f(x) \to \infty$	$f(x) \to \infty$
g	continuous	continuous	continuous	continuous	continuous	continuous	continuous
h	increases	increases	increases	increases	increases	increases	increases
i	1.2^x	1.4^x	1.7^x	2^x	e^x	3^x	10^x

2. The class of functions $f(x) = \log_b x \quad b > 1$ can be characterized as follows:

(a) The domain is all positive real numbers
(b) The range is all real numbers
(c) $f^{-1}(0) = 1$. The graph passes through the point (1,0)
(d) $f^{-1}(1) = b$. The graph passes through the point (b,1)
(e) $\lim_{x \to -\infty} f(x) = -\infty$
(f) $\lim_{x \to \infty} f(x) = +\infty$
(g) They are continuous throughout their domain
(h) As x increases, f(x) increases. f(x) is monotonic increasing.
(i)* $f^{-1}(\log_b x) = b^x$

Again we have been able to compile an impressive list of statements. Many of these statements will not be proved until the Calculus. Each statement is true.

---

*One easy way to tell inverses of functions on a graph is to reflect the graph along the line $f(x) = x$.

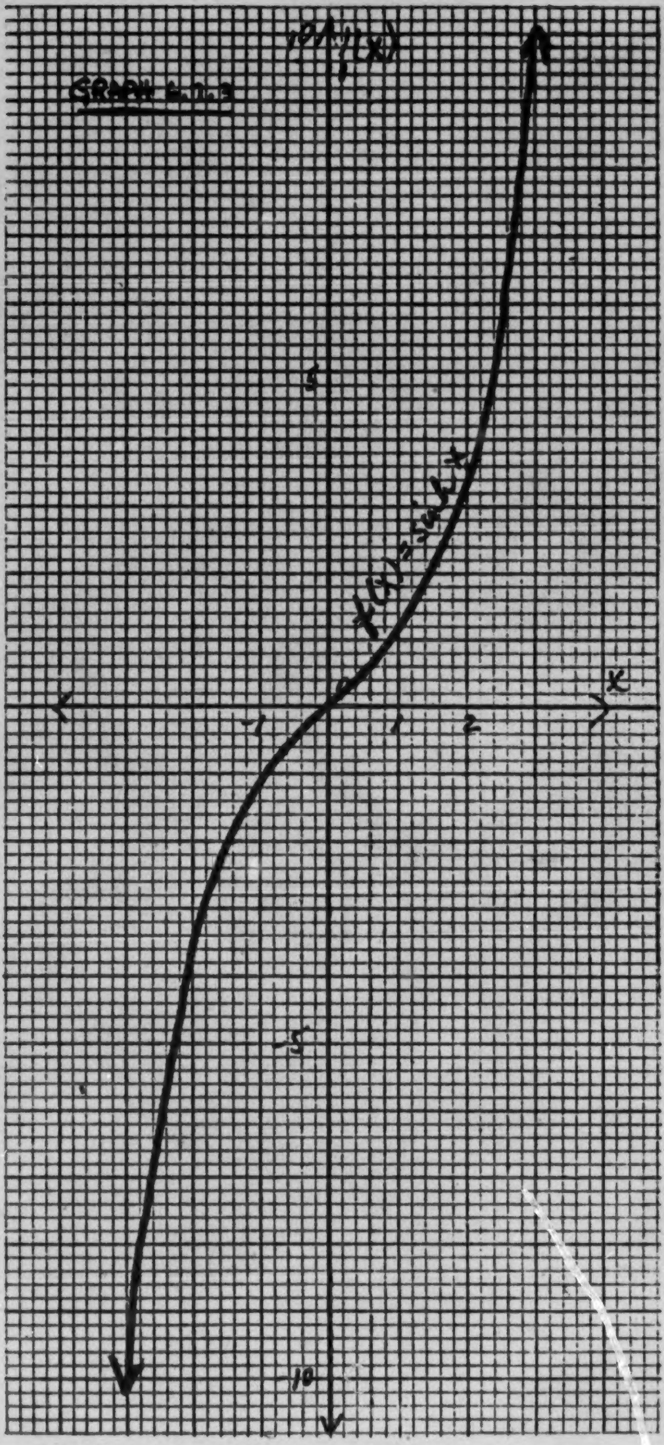
Several combinations of exponential functions appear often enough in physical applications of mathematics to be given special names. Two examples are the hyperbolic sine, denoted sinh and the hyperbolic cosine, denoted cosh.

$$\sinh x = \frac{e^x - e^{-x}}{2} .$$

$$\cosh x = \frac{e^x + e^{-x}}{2} .$$

EXAMPLE 6.7.3

Graph $f(x) = \sin h = \frac{e^x - e^{-x}}{2}$ in the interval $[-3, +3]$. Use the techniques from Section 5.1 and choose $d \leq .5$.



Exercise Set 6.7

(1 - 5) Consider the graphs in EXAMPLES 6.7.1 and 6.7.2. Answer each of the following with >, <, = or none of these.

1) If $1 < a < b$, $x > 0$ then a^x ______ b^x

2) If $1 < a < b$, $x > 0$ then $\log_a x$ ______ $\log_b x$

3) If $a, x > 1$ then a^x ______ $\log_a x$

4) If $0 < x < y$ then $\log_b x$ ______ $\log_b y$

5) If $0 < x < y$ then b^x ______ b^y

(6 - 8) Complete each of the following with <, > or =.

6) $\sinh x + \cosh x$ ______ e^x

7) $\cosh^2 x - \sinh^2 x$ ______ 1

8) $\cosh^2 x + \sinh^2 x$ ______ 1

9) graph $f(x) = \sinh$, $g(x) = \cosh$* and $h(x) = e^x$ on the same axis in the interval $[-3, +3]$. What relationship holds among these functions when $x > 0$?

(10 - 13) Consider the class of functions $f(x) = b^x$ where $0 < b < 1$. Some members of this class are:

$f(x) = .5^x$

$f(x) = .01^x$

$f(x) = \frac{1}{3}^x$

$f(x) = \frac{2}{7}^x$

---

*The shape of this graph, although it looks like a parabola, is called a catenary. It is not a parabola because it is not a polynomial.

10) Graph each of these functions and determine

- a) domain
- b) range
- c) $f(0)$
- d) $f(1)$
- e) the character of f $f(x)$ as $x \rightarrow -\infty$
- f) the character of $f(x)$ as $x \rightarrow +\infty$
- g) the continuity or discontinuity of $f(x)$
- h) $f^{-1}(x)$
- i) the character of $f(x)$ as x increases.

11) Make a list of generalizations about $f(x) = b^x$ when $0 < b < 1$.

12) Compare your list from exercise 11 to the list of EXAMPLE 6.7.1. Make a list of generalizations about $f(x) = b^x$ when $b > 0$, $b \neq 1$.

13) Graph $f(x) = b^x$ when $b = 1$. Does this fit your generalizations of exercise 11? Why is 1 not considered as a base for an exponential function?

(14 - 17) Consider the class of functions $f(x) = \log_b x$ where $0 < b < 1$. Some members of this class are:

$$f(x) = \log_{.5} x$$

$$f(x) = \log_{.01} x$$

$$f(x) = \log_{\frac{1}{3}} x$$

$$f(x) = \log_{\frac{2}{7}} x$$

14) Graph each of these functions and determine

- a) domain
- b) range
- c) $f^{-1}(0)$
- d) $f^{-1}(1)$
- e) the character of $f(x)$ as $x \to 0^+$
- f) the character of $f(x)$ as $x \to +\infty$
- g) the continuity or discontinuity of $f(x)$
- h) $f^{-1}(x)$
- i) the character of $f(x)$ as x increases.

15) Make a list of generalizations about $f(x) = \log_b x$ when $0 < b < 1$.

16) Compare your list from exercise 14 to the list of EXAMPLE 6.7.2. Make a list of generalizations about $f(x) = \log_b x$ when $b > 0$, $b \neq 1$.

17) Graph $f(x) = \log_b x$ when $b = 1$. Does this fit your generalizations of exercise 15? Why is 1 not considered as a base for a logarithmic function?

(18 - 19) Consider the class of functions $f(x) = b^x$ where $b < 0$. One member of this class is $f(x) = (-.5)^x$.

18) graph $f(x) = (-.5)^x$

19) What outstanding characteristics does this function have?

(20 - 21) Consider the functions:

$f(x) = \frac{1}{2}^{x}$

$g(x) = x^2$

$h(x) = 2^x$

$j(x) = x^{\frac{1}{2}}$

$k(x) = \log_2 x$

$\ell(x) = \log 2x$

$m(x) = 2 \log x$

$n(x) = \log_{\frac{1}{2}} x$

20) Make a sketch of each of these functions.

21) Which of these functions are inverses?

22) On the same set of axes, graph $f(x) = \ln|x|$ and $g(x) = |\ln x|$. Compare these two functions.

23) The function $f(x) = x^x$ has many interesting properties.

  a) Graph $f(x) = x^x$ when $x > 0$.

  b) Graph $f(x) = x^x$ when $x < 0$.

  c) Based on your answers to a) and b), what can you say about 0^0?

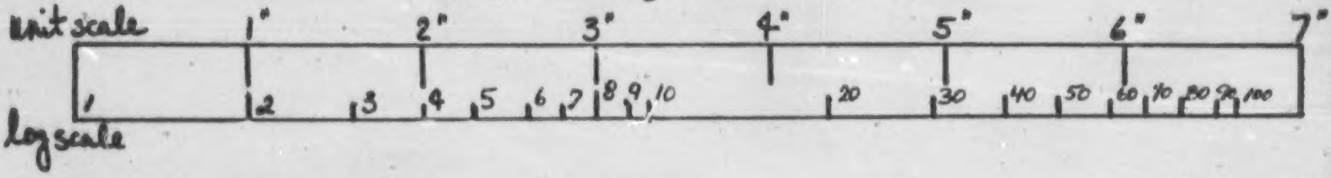
6.8 Logarithmic Scales

In this section we will consider several uses for logarithmic scales.

Ordinary graphs are made on paper that is divided into units by a uniform scale. Often it is more convenient, more revealing and more practical to use another kind of scale called a logarithmic scale. On a logarithmic scale the origin represents the value of the variable and the other convenient values of the variable are marked at distances from the origin corresponding to the logarithms.

EXAMPLE 6.8.1

Construct a logarithmic scale.



This scale is determined by the log of each of the numbers. Since logarithms can be determined for any appropriate base, log scales can be made to correspond to any different choice of unit length. This scale is made from a unit of one inch and base 2.

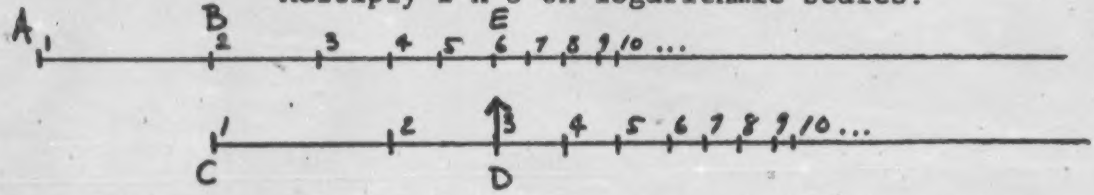
$\log_2 1 = 0 \longrightarrow 0$ inch
$\log_2 2 = 1 \longrightarrow 1$ inch
$\log_2 3 = 1.6 \longrightarrow 1.6$ inches
$\log_2 4 = 2 \longrightarrow 2$ inches
$\log_2 5 = 2.3 \longrightarrow 2.3$ inches

$\log_2 6 = 2.6 \longrightarrow 2.6$ inches
$\log_2 7 = 2.8 \longrightarrow 2.8$ inches
$\log_2 8 = 3. \longrightarrow 3$ inches
$\log_2 9 = 3.2 \longrightarrow 3.2$ inches
$\log_2 10 = 3.3 \longrightarrow 3.3$ inches
$\vdots$

Before the wide availability of calculators and computers many involved calculations were computed by a device called a slide rule. A slide rule uses the principles of logarithms and logarithmic scales. A simple slide rule can be constructed by making two logarithmic scales of the same base. Multiplication is carried out by the property that $\log xy = \log x + \log y$. The sum of the logarithms is given by the sum of the corresponding distances on the scale.

EXAMPLE 6.8.2

Multiply 2 x 3 on logarithmic scales.



$\log 2 = AB$

$\log 3 = CD$

$\log 2 + \log 3 = AB + CD = AE$ thus $2 \times 3 = 6$.

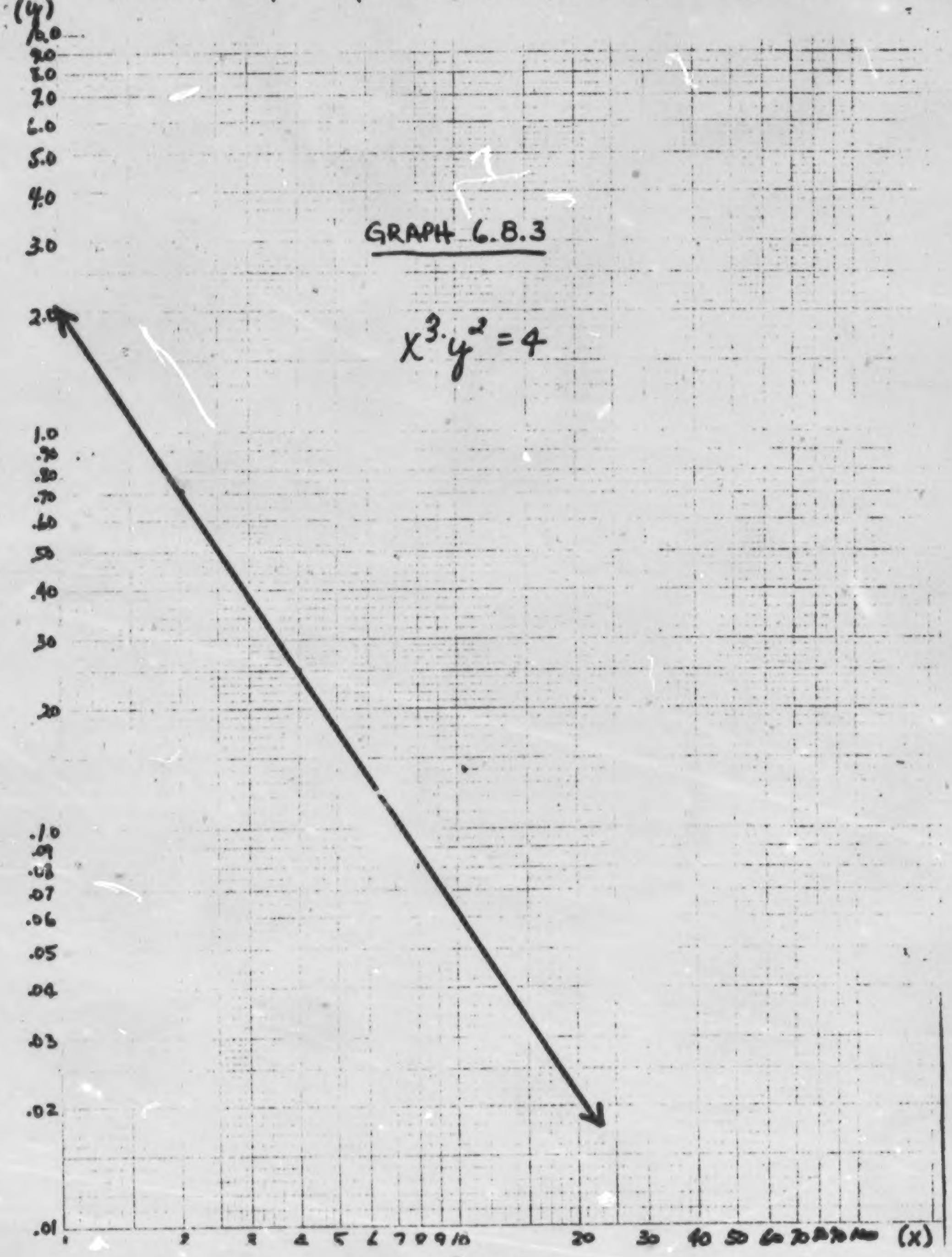
Just as for linear scales, there is graph paper having logarithmic scales. If both scales are logarithmic it is called full logarithmic paper. If one scale is logarithmic and one is linear it is called semilogarithmic paper. The principal advantage of logarithmic or semilogarithmic graphs is that certain functions have much simpler graphs on these kinds of paper than on linear scales. Because logs are only defined on positive numbers at most two quadrants, rather than four quadrants are necessary.

EXAMPLE 6.8.3

Graph $x^3y^2 = 4$ on full logarithmic paper.

A table of values is:

x	1	2	3	4	5	6	7	8	9	10
y	2	.707	.385	.250	.179	.136	.108	.098	.074	.063



GRAPH 6.8.3

$x^3 \cdot y^2 = 4$

This graph is a straight line because

$\log_b(x^3\ y^2) = \log_b 4$ becomes

$3\log_b x + 2\log_b y = \log_b 4$ which is a linear

equation in $\log_b x$ and $\log_b y$.

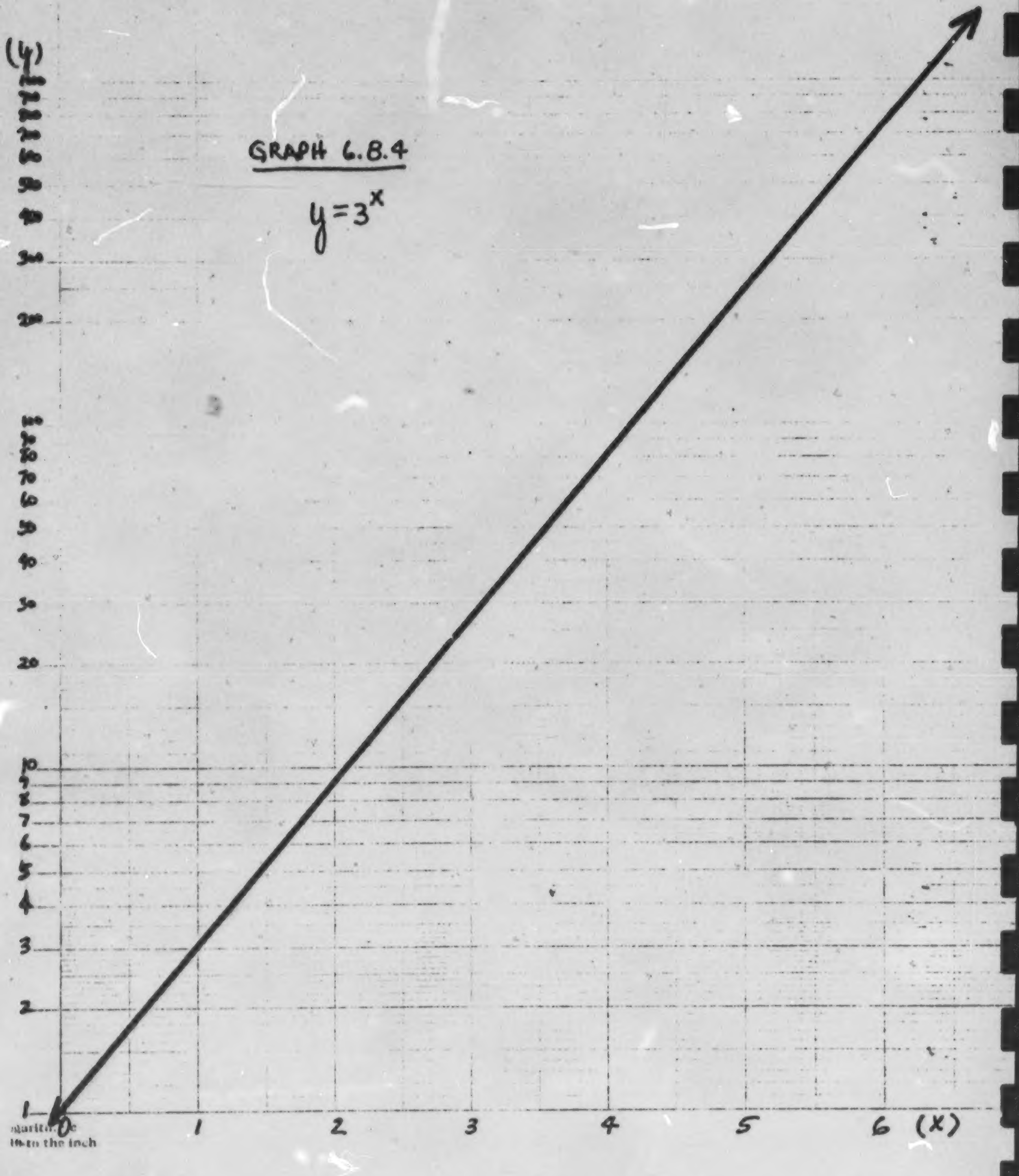
EXAMPLE 6.8.4

Graph $y = 3^x$ on semilogarithmic paper.

$\log_b y = x(\log_b 3)$

Since $\log_b 3$ is a constant it is convenient to use a log scale for y and a linear scale for x. A table of values is:

x	0	1	2	3	4	5	6
y	1	3	9	27	81	243	729


GRAPH 6.8.4
$y=3^x$
(y)
1
2
3
4
5
6
7
8
9
10
20
30
40
50
100
200
300
0
1
2
3
4
5
6
(x)

Exercise Set for 6.8

(1 - 10) Build a slide rule using semilog graph paper and use it to compute each of the following:

1) 2 x 8
2) 3 x 6
3) 5.3 x 6
4) 4.2 x 3
5) 2.7 x 1.5
6) 5.1 x 2.3
7) $(5)^2$
8) $(7)^2$
9) $(1.8)^3$
10) $(2.9)^3$

(11 - 14) Develop a way to divide on a slide rule. Use your "homemade" slide rule to determine:

11) 50 ÷ 2
12) 18 ÷ 3
13) 3.6 ÷ 1.2
14) 4.8 ÷ 1.6

(15 - 18) Graph each of the following on full log paper.

15) $xy = 3$
16) $xy^3 = 4$
17) $xy^2 = 1.7$
18) $x^2y^3 = 3.1$

(19 - 22) Graph each of the following on semi-log paper.

(19) $y = 1.7^x$
20) $y = 2.5^x$
(21) $y = 2^{3x}$
22) $y = 3^{1.1x}$

(23 - 28) Identify the scales on which each of the following would be a straight line.

23) $y = x^3$
24) $y^2 = x^3$
25) $x^3 = y^{1.2}$
26) $y = 5x^3$
27) $y = 7^{2x}$
28) $y = 2^x$

6.9 Chapter 6 Test

(1-2) Evaluate to 4 decimal places:

1) $\sqrt[5]{-7}$ 2) $\log_3 5$

3) Determine the exact value of 31^8.

4) Determine the value of

$$\frac{\log_{.8} \frac{16}{25} \div \log_2\sqrt{2}}{\log_4 16 - \log_3 9\sqrt{3}}$$

5) Determine the exact value of $\lim\limits_{n\to\infty}(1 + \frac{1}{n})^n$.

6) Solve for n: $S = ar^{n-1}$

(7-8) Solve for x to 4 decimal places:

7) $5^{x+2} = 3^{x-2}$ 8) $\log \ln (x-3) = -.2$

(9-10) Identify the scales on which each of the following would be a straight line.

9) $y = x^2$ 10) $y = 2^x$

11) Assume that it is known that the number of bacteria, N, present after t minutes is given by

$$N = 1000e^{0.25t}$$

a) How many bacteria are present after one-half hour?

b) How long will it take to have 50,000 bacteria?

(12-15) Complete each of the following:

12) The domain of the function $f(x) = \log_{2.1} x$ is _____.

13) If $f(x) = \log_b x$ then $f^{-1}(1) =$ _____.

14) If $f(x) = \log_b x$ then $f^{-1}(x) =$ _____.

15) If $f(x) = 1.3^x$ then $\lim_{n \to -\infty} f(x) =$ _____.

(16-20) Decide whether each of the following is true or false.

16) $\ln x^p = \ln p x$

17) $10^{\log x} = x$

18) The natural log of a negative number is a negative number.

19) If $0 < y < x$ and $b > 1$ then $b^y < b^x$.

20) $\ln |x| = |\ln x|$

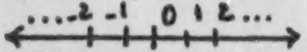
CHAPTER 7. TRIGONOMETRIC FUNCTIONS

In this chapter we will discuss some familiar ideas in new settings and some new ideas in familiar settings. In particular we will study the amazing relationships between trigonometric functions and complex numbers.

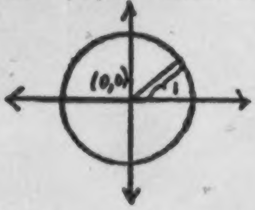
7.1 Circular Functions

We wish to create a function that associates the real numbers with the points on a circle. We will call this function, C, for circular. The domain of C is all real numbers and the range of C is the points on a circle. For easy reference let's locate our circle on the coordinate plane having center (0,0) and radius 1.

C:



all real numbers



To have a place to start let C(1) = (1,0). The other functional values of C can be determined by "wrapping" the number line around the unit circle. The circumference of the unit circle is 2π - so some functional values of C are easy to determine.

$$C(\pi/2) = (0,1)$$
$$C(\pi) = (-1,0)$$
$$C(\tfrac{3\pi}{2}) = (0,-1)$$
$$C(2\pi) = (0,1)$$

One interesting property of C is already apparent.

$$C(0) = C(2\pi) = C(-2\pi) = C(4\pi) = C(-4\pi) = \dots.$$

C is clearly not a 1 - 1 function. Infinitely many real numbers are associated with the same point on the unit circle.

$$C(t + 2\pi) = C(t) \text{ for any real number } t.$$

Definition 7.1.1 A function which has the property that it repeats in value at regular intervals is called a circular function (or periodic function or wrapping function).

Our function C repeats at the regular interval 2π so it is said to have period 2π. Another interesting property of C is that since it is not 1 - 1, it cannot have an inverse.

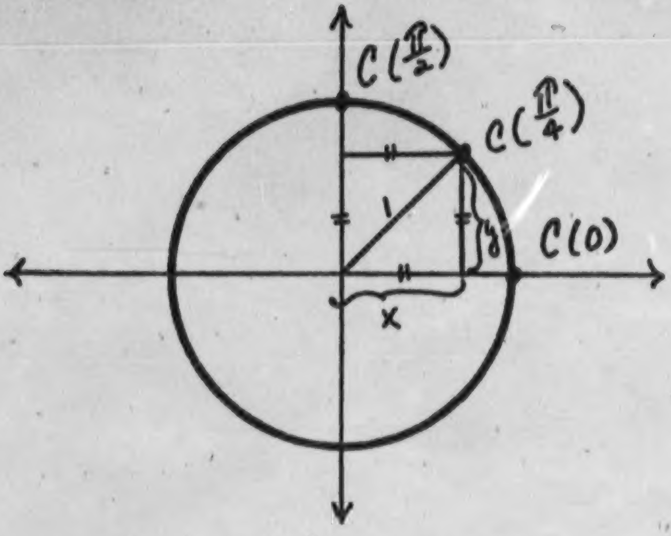
Let us determine more of the functional values of C. Our knowledge of the circle and analytic geometry can be brought to bear in our analysis.

EXAMPLE 7.1.2 What is $C(\frac{\pi}{4})$?

Solution: $C(0) = (1,0)$

$C(\frac{\pi}{2}) = (0,1)$

$C(\frac{\pi}{4}) = ?$


$C(\frac{\pi}{2})$
$C(\frac{\pi}{4})$
1
y
$C(0)$
x

$C(\frac{\pi}{4}) = (x, y)$ where x and y are the sides of a square whose diagonal is 1

so $\quad x^2 + y^2 = 1$

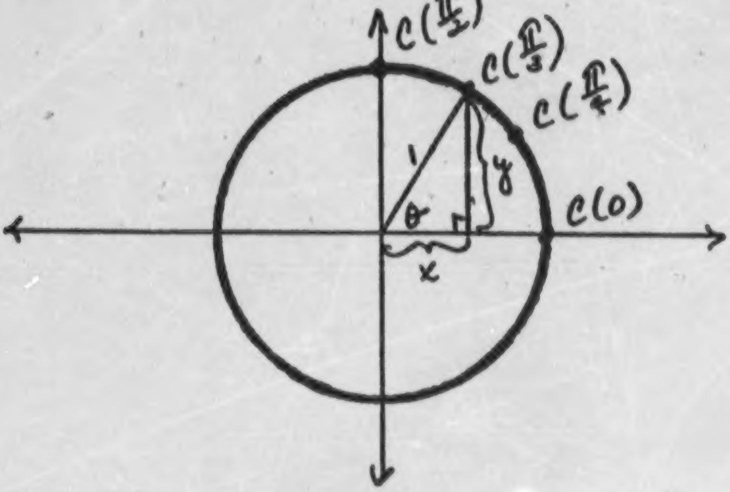
$$x^2 + x^2 = 1$$

$$2x^2 = 1$$

$$x = \frac{\sqrt{2}}{2} = y$$

thus $C(\frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \doteq (.7071, .7071)$.

EXAMPLE 7.1.3 What is $C(\frac{\pi}{3})$?


$C(\frac{\pi}{2})$
$C(\frac{\pi}{3})$
$C(\frac{\pi}{4})$
1
y
θ
$C(0)$
x

$\frac{\pi}{3} = \frac{1}{6}(2\pi)$ so $\frac{\pi}{3}$ is $\frac{1}{6}$ of the way around the circle starting at (1,0) so $\theta = \frac{1}{6}(360^\circ) = 60^\circ$ and $x = .5$,

$y = .5\sqrt{3}$ so

$$C\left(\frac{\pi}{3}\right) \doteq (.5,\ .8660).$$

EXAMPLE 7.1.4 What is C(1)?

Solution: $\frac{\pi}{4} \doteq .7854 < 1 < \frac{\pi}{3} \doteq 1.0472$

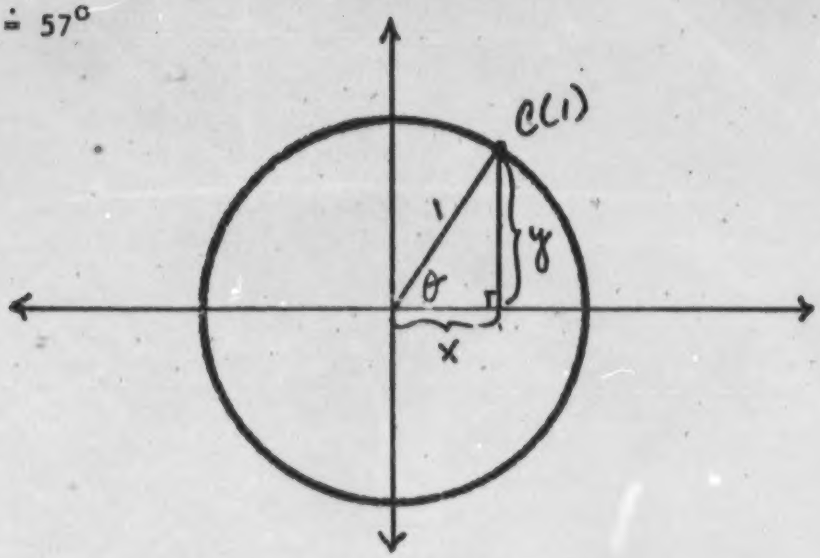
$$\tfrac{1}{2}\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \tfrac{1}{2}\left(\frac{7\pi}{12}\right) = \frac{7\pi}{24} \doteq .9163 \quad \text{too small}$$

$$\tfrac{1}{2}\left(\frac{7\pi}{24} + \frac{\pi}{3}\right) = \tfrac{1}{2}\left(\frac{15\pi}{24}\right) = \frac{15\pi}{48} \doteq .9817 \quad \text{too small}$$

$$\tfrac{1}{2}\left(\frac{15\pi}{48} + \frac{\pi}{3}\right) = \tfrac{1}{2}\left(\frac{31\pi}{48}\right) = \frac{31\pi}{96} \doteq 1.0145 \quad \text{too big}$$

$$\tfrac{1}{2}\left(\frac{15\pi}{48} + \frac{31\pi}{96}\right) = \tfrac{1}{2}\left(\frac{61\pi}{96}\right) = \frac{61\pi}{192} \doteq .9981 \quad \text{close enough.}$$

$.9981 \doteq (.1589)\,2\pi$ so 1 is approximately .1589 of the way around the circle starting at (1,0) so $\theta = .1589(360^\circ) \doteq 57^\circ$



$x \doteq \cos 57^\circ$, $y \doteq \sin 57^\circ$ so $C(1) \doteq (.5446,\ .8387)$.

A better approximation of C(1) is (.5403, .8415). Can you tell why? Pause to think for a minute before you continue reading.

We have really been discussing the familiar concept of radians. Thus we can make the following reasonable definition of C.

Definition 7.1.5

Let t be a real number and C a circular function of period 2π then

$C(t) = (x, y)$ where

$x = \cos t$, $y = \sin t$ and t is in radian measure.

EXAMPLE 7.1.6

(a) Determine C(2).

$C(2) = (-.4161, .9093)$

(b) Determine all t so that $C(t) = (.8660, .5)$

$\mathrm{Cos}^{-1}(.8660) = 0.5236$

$\mathrm{Sin}^{-1}(.5) = 0.5236$

Thus $t \doteq .5236 + k\,2\pi$ where k is an integer.

Notice that .5236 radians $\equiv 30^{\circ}$ *

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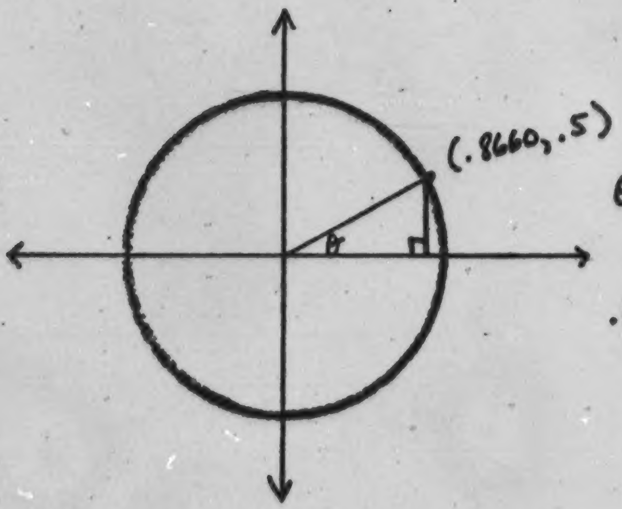
*The symbol $\equiv$ means equivalent.



$\theta = .5236$ radians

$.5 = \frac{1}{2}$

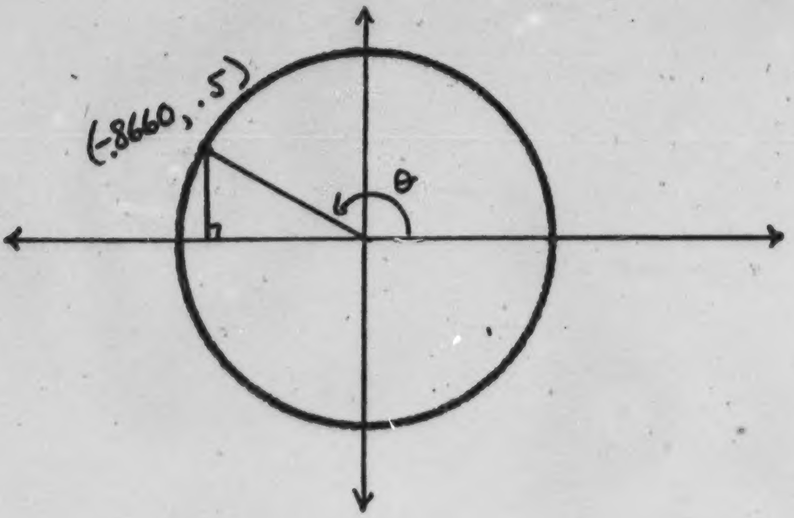
$.8660 \doteq \frac{\sqrt{3}}{2}$

(c) Determine all t so that $C(t) = (-.8660, .5)$

$\text{Cos}^{-1}(-.8660) = 2.6179$

$\text{Sin}^{-1}(.5) = .5236$

$2.6179 \neq .5236$ What could be wrong?



Recall that the $\text{Cos}^{-1}(x)$ and $\text{Sin}^{-1}(x)$ functions deal only with principal values of $\cos^{-1}(x)$ and $\sin^{-1}(x)$ relations.

now $\frac{\pi}{2} < t < \pi$ for t on the "first wrap" .

yet $.5236 < \frac{\pi}{2}$ but $\frac{\pi}{2} < 2.6179 < \pi$.

Recall that $\sin \theta = \sin (180 - \theta)$ (in degrees)

so $\sin \theta = \sin (\pi - \theta)$ (in radians)

so $\sin (.5236) = \sin (2.6179)$

and $t = 2.6179 + k2\pi$ where k is an integer.

Exercise Set 7.1

(1 - 6) For each of the following determine two elements in the range of C that are equal to the given point, where C is a circular function of period 2π .

Example:

$$C(3) = C(2\pi + 3) = D(3 - 6\pi)$$

1) $C(7)$

2) $C(5)$

3) $C(5\pi)$

4) $C(3\pi)$

5) $C(-.3573)$

6) $C(-1.5826)$

(7 - 12) For each of the following determine two elements in the domain of C^* where C^* is a circular function with period 4.

7) $C^*(1.2)$

8) $C^*(-3.7)$

9) $C^*(-\pi/6)$

10) $C^*(\sqrt{2})$

11) $C^*(e)$

12) $C^*(-e/\pi)$

(13 - 20) For each of the following let C be a circular function from the real numbers to the unit circle with $C(0) = (1,0)$. Determine:

13) $C(\frac{\pi}{2})$ 14) $C(\frac{3\pi}{2})$

15) $C(-\frac{5\pi}{6})$ 16) $C(-\frac{3\pi}{4})$

17) $C(5)$ 18) $C(10)$

19) $C(\sqrt{2})$ 20) $C(e)$

(21 - 26) For each of the following determine all t so that C(t) is the indicated-ordered pair. $C: \mathbb{R} \rightarrow$ unit circle such that $C(0) = (1,0)$. Express t in radians and degrees.

21) (.7071, .7071) 22) (.5, .8660)

23) $(\sqrt{5}/3, -2/3)$ 24) $(\sqrt{6}/3, \sqrt{3}/3)$

25) (-.6, -.8) 26) (-.4, -.9165)

(27 - 30) In each of the following, P is a periodic function such that $P(0) = (x,0)$ where $x > 0$.

(a) Find P(0)

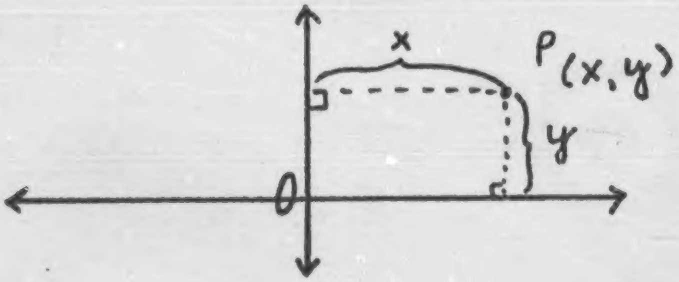
(b) Determine the period of each function P.

27) $P(.6435) = (3,4)$ 28) $P(3\pi) = (-3,0)$

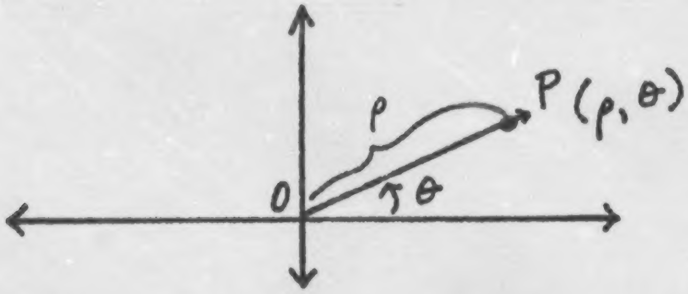
29) $P(-2.8176) = (-1, -2.9782)$ 30) $P(2.1112) = (5, -3)$

7.2 Polar Coordinates

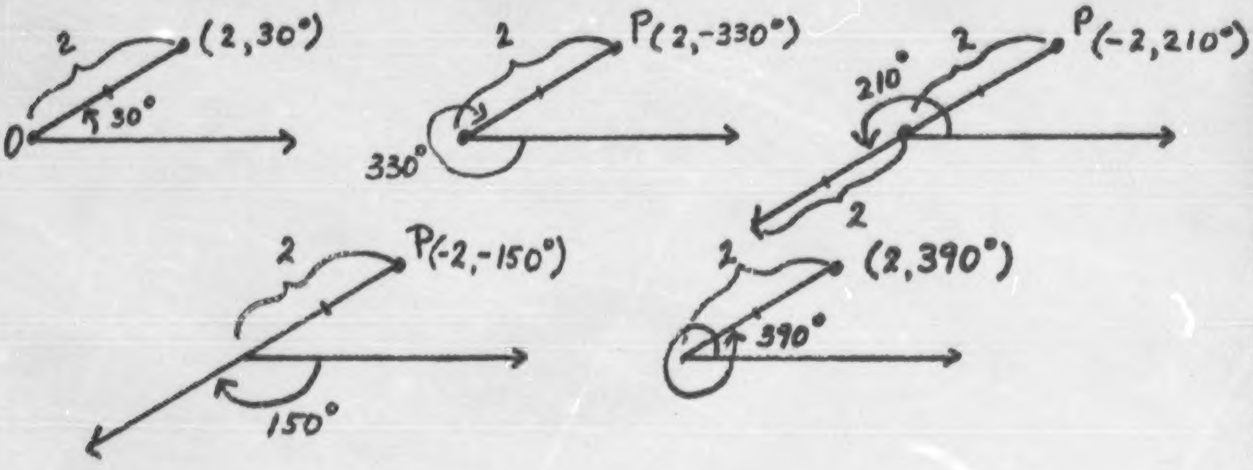
You are familiar with the idea of associating points in the plane with ordered pairs of real numbers. The rectangular or Euclidean or Cartesian plane is a grid in which the ordered pair associated with a point is determined by its directed distance, vertically and horizontally, from two perpendicular lines. Thus the ordered pair (x,y) represents a point, P, that is x distance vertically and y distance horizontally from these lines.



Another system of locating P is by measuring its distance from the origin and the angle that the ray $\overrightarrow{OP}$ makes with the positive direction of the vertical axis.

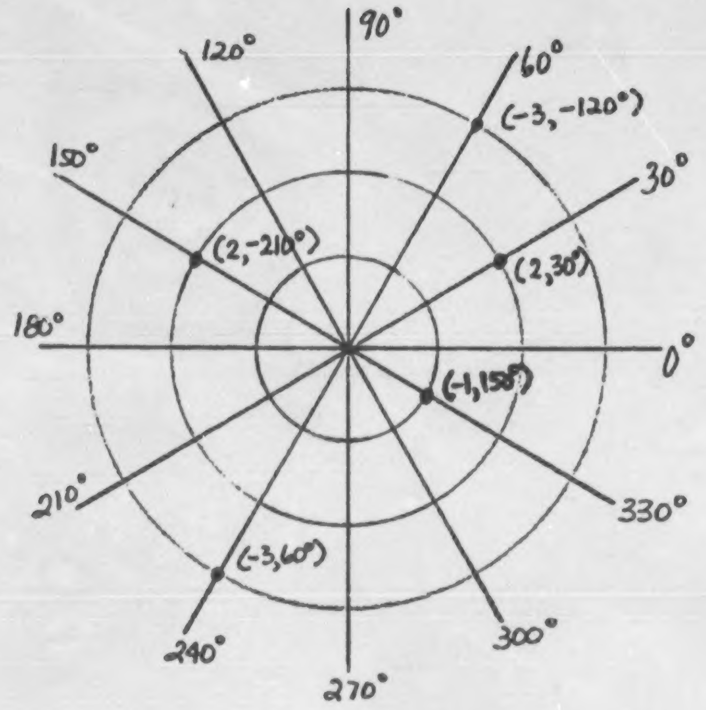


The segment OP is designed by ρ (rho, pronounced rō) and the angle θ (theta, pronounced thā′ta). Thus P, in polar coordinates, is represented as (ρ , θ). The origin, O is called the pole. The distance ρ may be any real number and is measured along the terminal side of θ . If ρ is negative it is measured along the terminal side of θ extended through the pole. θ is measured counterclockwise if it is positive and clockwise if it is negative. The polar coordinates (2, 30°), (2, -330°), (-2, 210°), (-2, -150°) and (2, 390°) all represent the same point.

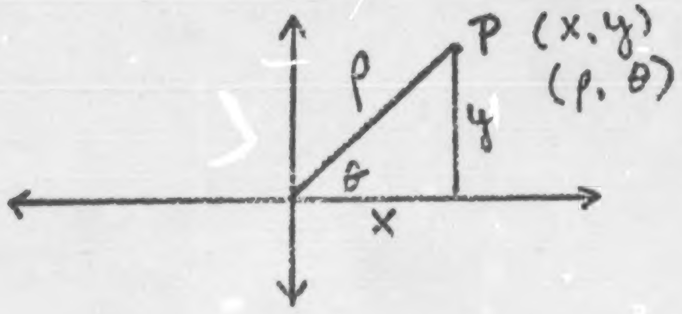


Clearly, there is no 1 - 1 correspondence of ordered pairs and points in the polar plane.

Points can be plotted conveniently on polar coordinate paper that has equally spaced concentric circles whose center is the pole and equally spaced radial lines through the pole.


90°
120°
60°
(-3, -120°)
150°
30°
(2, -210°)
(2, 30°)
180°
0°
(-1, 150°)
210°
330°
(-3, 60°)
240°
300°
270°

It is often convenient to switch from rectangular coordinates to polar coordinates or vice versa.



From trigonometry it is clear that

$$\cos\theta = \frac{x}{\rho}, \quad \sin\theta = \frac{y}{\rho}, \quad \tan\theta = \frac{y}{x}$$

$$\rho^2 = x^2 + y^2$$

and

$$\boxed{\begin{array}{ll} x = \rho\cdot\cos\theta\ , & y = \rho\cdot\sin\theta \\ \rho = \pm\sqrt{x^2 + y^2}, & \theta = \text{arc tan}\,\frac{y}{x} \end{array}} \tag{7.2.1}$$

These four equations allow us to easily change from one system to another. Most scientific calculators have special keys that allow you to perform a change with only a few keystrokes. Microprocessors can be easily programmed to execute these conversions.

EXAMPLE 7.2.2 Determine how to change the rectangular coordinate (x,y) to the polar coordinate (ρ , θ)

HP33E	TI-58	TRS-80
(key y)	(key x)	
ENTER	x >< t	10 INPUT X:Y
(key x)	(key y)	20 A = SQR(X↑2 + Y↑2)
g → P	INV 2nd P → R	30 B = (TAN(Y/X))*57.29[illegible]8
(display is ρ)	(display is θ)	40 PRINT "("A","B")"
R↓	x >< t	
(display is θ)	(display is ρ)	

Exercise Set 7.2

(1 - 10) Plot each of the following polar coordinates on polar coordinate paper.

1) $A(3, -30^\circ)$
2) $B\ (6, 60^\circ)$
3) $C(0, 20^\circ)$
4) $D\ (-1, 180^\circ)$
5) $E(4, 0^\circ)$
6) $F(4, \frac{\pi}{4})$
7) $G\ (+5, -\frac{2\pi}{3})$
8) $H\ (3, \frac{3\pi}{2})$
9) $I(-2, \frac{\pi}{2})$
10) $J\ (-6, -225^\circ)$

(11 - 14) Characterize the points where

11) $\rho = 2$
12) $\rho = -1$
13) $\theta = 45^\circ$
14) $\theta = \pi$

15) Determine how to change the polar coordinate (ρ, θ) to the rectangular coordinate (x,y) on your calculating device.

(16 - 21) Find the nonnegative polar coordinates for the following points (a) by hand

(b) by using your calculating device.

16) $(0, 2)$
17) $(-1, 0)$
18) $(-\sqrt{2}, \sqrt{2})$
19) $(2\sqrt{3}, -2)$
20) $(-4, -3)$
21) $(-5, -12)$

(22 - 27) Find the rectangular coordinates for each of the following points (a) by hand

(b) by using your calculating device

22) $(6, 270^{\circ})$

23) $(-8, 150^{\circ})$

24) $(3\sqrt{2}, 135^{\circ})$

25) $(2, -30^{\circ})$

26) $(-.5, -\frac{\pi}{2})$

27) $(-2, \frac{\pi}{6})$

28) Explain the relation between polar coordinates and the circular functions of section 7.1.

(29 - 30) Confer section 7.1 and use your knowledge of polar coordinates to find:

29) $C(5)$

30) t such that $C(t) = (.5, \frac{\sqrt{3}}{2})$

7.3 Graphs of Equations in Polar Coordinates

We have transformed points from rectangular coordinates to polar coordinates and the reverse. We can do the same for equations. Recall the equations from Section 7.2:

$$x = \rho \cos \theta \qquad y = \rho \sin \theta \qquad x^2 + y^2 = \rho^2$$

$$\rho = \pm \sqrt{x^2 + y^2} \qquad \theta = \arctan \frac{y}{x}$$

EXAMPLE 7.3.1 Find the polar coordinate equation corresponding to $3x - 2y = 2$.

Solution: $3x - 2y = 2$

$3(\rho \cos \theta) - 2(\rho \sin \theta) = 2$ by substitution

$3 \rho \cos \theta - 2 \rho \sin \theta = 2$

$\rho (3 \cos \theta - 2 \sin \theta) = 2$

EXAMPLE 7.3.2 Find the rectangular coordinate equation corresponding to $\rho = \frac{3}{1 + 4 \cos \theta}$

Solution: $\rho = \frac{3}{1 + 4 \cos \theta}$

$\rho + 4 \rho \cos \theta = 3$

$\rho + 4x = 3$

$\rho = 3 - 4x$

$\rho^2 = 9 - 24x + 16x^2$

$x^2 + y^2 = 9 - 24x + 16x^2$

$y^2 = 9 - 24x + 15x^2$

We can also graph polar equations.

EXAMPLE 7.3.3 Graph the polar equation

$$\rho = 2(1 + \sin\theta)$$

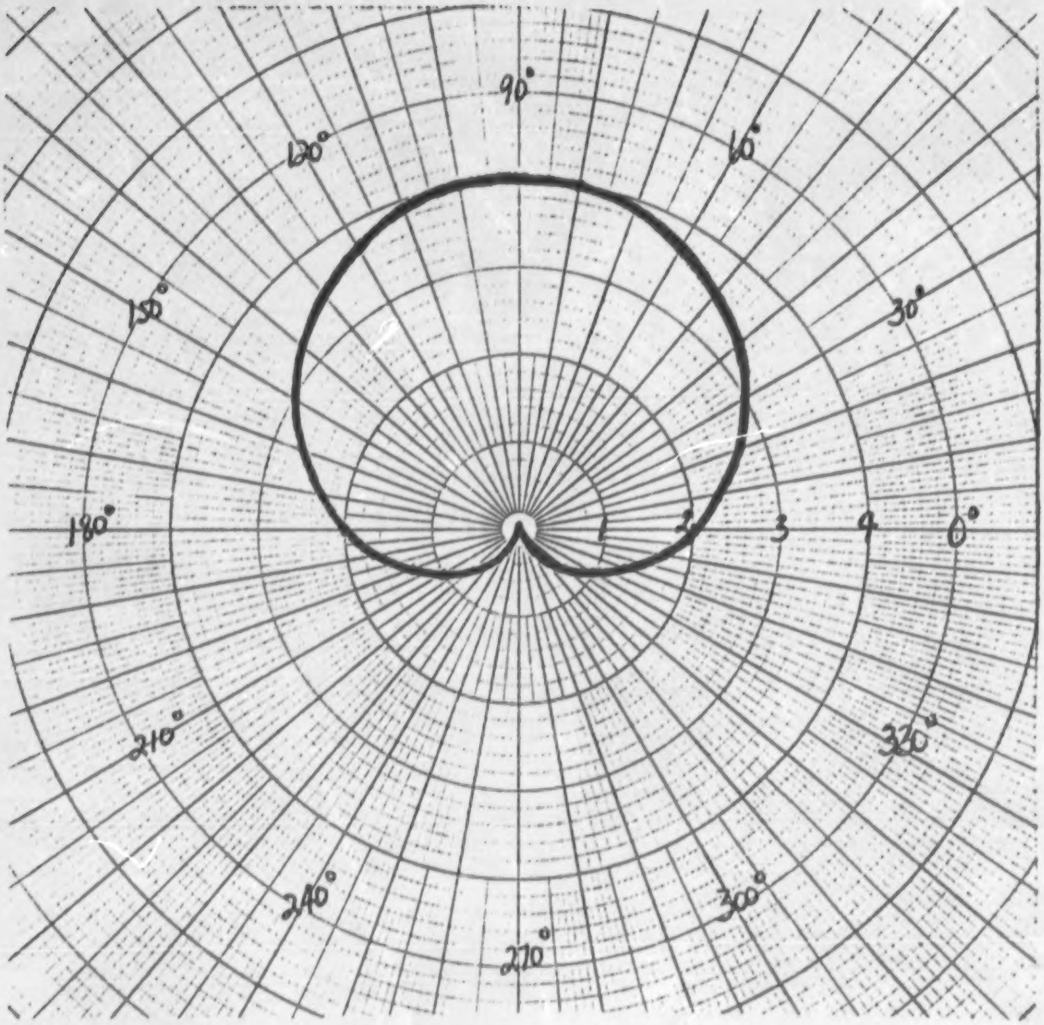
A program is helpful in determining coordinates.

θ	0°	10°	20°	30°	40°	50°	60°	70°	80°	9
ρ	2	2.3	2.7	3	3.3	3.5	3.7	3.9	4.0	4

θ	100°	110°	120°	130°	140°	150°	160°	170°	180°
ρ	4	3.9	3.7	3.5	3.3	3	2.7	2.3	2

θ	190°	200°	210°	220°	230°	240°	250°	260°	270°
ρ	1.7	1.3	1	.7	.5	.3	.1	.03	0

θ	280°	290°	300°	310°	320°	330°	340°	350°	360°
ρ	.03	.1	.3	.5	.7	1	1.3	1.7	2



This graph is called a cardioid because it is a heart shaped curve.

Exercise Set 7.3

(1 - 10) Transform the following equations to the corresponding polar coordinate equations.

1) $x = 4$ 2) $y = -3$

3) $2x - 3y = 7$ 4) $x = y + 4$

5) $x^2 + y^2 = 9$ 6) $x^2y = 10$

7) $xy = 7$ 8) $x^2 + y^2 - 3x + 2y = 0$

9) $x^2 - y^2 = 10$ 10) $y^2 = 3x$

(11-20) Transform the following equations to the corresponding rectangular coordinate equation.

11) $\rho = 3$ 12) $\theta = 30^\circ$

13) $\rho = 3 \sin \theta + 3 \cos \theta$ 14) $\rho = 3 \sin \theta - 2 \cos \theta$

15) $\rho \cos \theta = 8$ 16) $\rho^3 \sin \theta = 9$

17) $\rho = \frac{2}{1 + 3 \cos \theta}$ 18) $\rho = \frac{3}{\sin \theta + 2 \cos \theta}$

19) $\rho = \sec \theta$ 20) $\rho = \tan \theta$

(21 - 34) Graph each of the following on polar coordinate paper. Plot points at 10° intervals.

21) $\rho = 1 + 2 \sin \theta$ 22) $\rho = 1 - \cos \theta$

23) $\rho = 3 \cos 2\theta$ 24) $\rho = 2 \sin 3\theta$

25) $\rho = \tan \theta$ 26) $\rho = \sec \theta$

27) $\rho = \frac{5}{3 + 2 \sin \theta}$ 28) $\rho = \frac{2}{1 - \cos \theta}$

29) $\rho = 2 + 2 \cos \theta$ 30) $\rho = 3 + 2 \sin \theta$

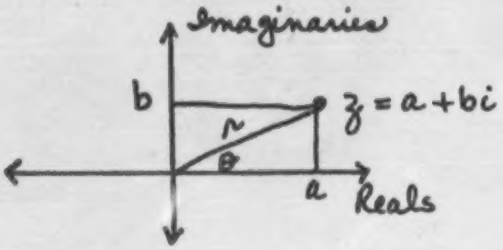
31) $\rho = 3\theta$ 32) $\rho\theta = 6$

33) $\rho = e^{2\theta}$ 34) $\rho = 4 \sec \theta$

7.4 Polar Form of Complex Numbers

Complex numbers can be graphed by using a coordinate system having a real and an imaginary axis. We can also graph complex numbers in polar coordinates.

Let $z = a + bi$ then



$$\frac{a}{r} = \cos\theta$$

$$\frac{b}{r} = \sin\theta$$

$a = r\cos\theta$ and $b = r\sin\theta$. Thus $z = r\cos\theta + r\sin\theta$ or $z = r(\cos\theta + i\sin\theta)$ and explain the following important definition.

Definition 7.4.1. Let $z = a + bi$ be a complex number then $z = r(\cos\theta + i\sin\theta)$. This is called the polar form of a complex number. The length r is called the modulus of z and the angle θ is called the argument of z.

Since r is a length it is always nonnegative. In fact r is the absolute value of z or $r = |z|$. Recall that $|x| = \sqrt{x^2}$, so $r = |z| = \sqrt{(a + bi)^2} = \sqrt{a^2 + b^2}$.

EXAMPLE 7.4.2 Express $-2 + 2i\sqrt{3}$ in polar form

$$r = \sqrt{(-2)^2 + (2\sqrt{3})^2}$$

$$= \sqrt{4 + 12}$$

$$= 4$$

$$\cos\theta = \frac{-2}{4} \qquad \sin\theta = \frac{2\sqrt{3}}{4}$$

$$\cos\theta = -\frac{1}{2} \qquad \sin\theta = \frac{\sqrt{3}}{2}$$

$$\theta = 120^\circ \qquad \theta = 60^\circ$$

θ must be a 2nd quadrant angle.

$\sin 60^\circ = \sin 120^\circ$ so $\theta = 120^\circ$.

$$-2 + 2i\sqrt{3} = 4(\cos 120^\circ + i \sin 120^\circ)$$

This example exhibits the need to remember the following identities.

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

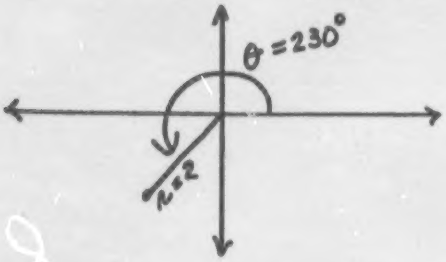
$$\sin(180 - \theta)^\circ = \sin\theta \qquad \cos(180 - \theta)^\circ = -\cos\theta$$

The following verbal algorithm can be used to write a program to do this example.

```
1. remember a, b
2. $r \leftarrow \sqrt{a^2 + b^2}$, remember r
3. $\theta \leftarrow \cos^{-1}(\frac{a}{r})$, remember $\theta$
4. $\theta^* \leftarrow \sin^{-1}(\frac{b}{r})$, remember $\theta^*$
5. display r, $\theta$, $\theta^*$
6. stop
```

θ and θ^* are both included in this algorithm to help us remember that $\cos^{-1}x$ and $\sin^{-1}x$ are not functions and we may have to think carefully before we can decide the value of θ .

EXAMPLE 7.4.3 Express $2(\cos 230^\circ + i \sin 230^\circ)$ in standard algebraic form.



$a = 2(\cos 230^\circ) \doteq -1.2856$.

$b = 2(\sin 230^\circ) \doteq -1.5321$

$2(\cos 230^\circ + i \sin 230^\circ) \doteq -1.2856 - 1.5321i$

The following algorithm can be used to write a program to do this example.

```
1. Remember r, $\theta$
2. a ← r cos $\theta$ , remember a
3. b ← r sin $\theta$ , remember b
4. display a, b
5. stop.
```

The polar forms of complex numbers enable us to easily multiply and divide them.

Theorem 7.4.5 If $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ then

$$z_1 z_2 = r_1 \cdot r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\right]$$

Proof

$$z_1 z_2 = \left[r_1(\cos \theta_1 + i \sin \theta_1)\right] \cdot \left[r_2(\cos \theta_2 + i \sin \theta_2)\right]$$

$$= r_1 r_2 \left[\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta\right.$$

$$\left. + i^2 \sin \theta_1 \sin \theta_2\right]$$

Recall that:

$$\cos x \cos y - \sin x \sin y = \cos (x+y)$$

$$\sin x \cos y + \cos x \sin y = \sin (x+y)$$

$$z_1z_2 = r_1r_2 \left[\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i (\sin \theta_1 \cos \theta_2 + \cos \theta_2 \sin \theta_1)\right]$$

and

$$z_1z_2 = r_1r_2 \left[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)\right]$$

EXAMPLE 7.4.6 Find the product

$$3(\cos 32^\circ + i \sin 32^\circ) \cdot 5(\cos 106^\circ + i \sin 106^\circ)$$
$$= 3(5) \left[\cos (32 + 106)^\circ + i \sin (32 + 106)^\circ\right]$$
$$= 15 (\cos 138^\circ + i \sin 138^\circ)$$

Theorem 7.4.7

If $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$ and

$z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$ then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \left[\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)\right].$$

Proof

$$\frac{z_1}{z_2} = \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2 (\cos \theta_2 + i \sin \theta_2)}$$

$$= \frac{r_1}{r_2} \left(\frac{\cos \theta_1 + i \sin \theta_1)}{\cos \theta_2 + i \sin \theta_2)} \cdot \frac{(\cos \theta_2 - i \sin \theta_2)}{(\cos \theta_2 - i \sin \theta_2)}\right.$$

$$= \frac{r_1}{r_2}\,\frac{(\cos\theta_1\cos\theta_2 - i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2)}{(\cos^2\theta_2 + \sin^2\theta_2)}$$

$$= \frac{r_1}{r_2}\,\frac{[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)]}{(\cos^2\theta_2 + \sin^2\theta_2)}$$

Recall that

$$\cos x\cos y + \sin x\sin y = \cos(x-y)$$

$$\sin x\cos y - \sin y\cos x = \sin(x-y)$$

$$\sin^2 x + \cos^2 x = 1$$

thus

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

EXAMPLE 7.4.8 Find the quotient

$$5(\cos 120^\circ + i\sin 120^\circ) \div 2(\cos 212^\circ + i\sin 212^\circ)$$

$$= \frac{5}{2}\left[\cos(120 - 212)^\circ + i\sin(120 - 212)^\circ\right]$$

$$= 2.5\left[\cos(-92)^\circ + i\sin(-92)^\circ\right]$$

$$= 2.5\,(\cos 268^\circ + i\sin 268^\circ)$$

It is customary to represent the polar forms of complex numbers in terms of positive angles.

Exercise Set 7.4

(1 - 8) Rewrite each of the following complex numbers in polar form.[*]

1) $2 - 3i$ 2) $-5 + 3i$

3) $3\sqrt{3} - 3i$ 4) $1 + i\sqrt{3}$

5) -1 6) 6

7) $5i$ 8) $-3i$

(9 - 16) Rewrite each of the following polar forms of complex numbers in standard algebraic form.[*]

9) $5(\cos 30^\circ + i \sin 30^\circ)$ 10) $3(\cos 210^\circ + i \sin 210^\circ)$

11) $3(\cos 90^\circ + i \sin 90^\circ)$ 12) $\cos 0^\circ + i \sin 0^\circ$

13) $.73(\cos 118^\circ + i \sin 118^\circ)$ 14) $3.42(\cos 243^\circ + i \sin 243^\circ)$

15) $\sqrt{3}(\cos 275^\circ + i \sin 275^\circ)$ 16) $\sqrt{7}(\cos 138^\circ + i \sin 138^\circ)$

(17 - 24) Perform the following operations.

17) $3 \cos 10^\circ + i \sin 10^\circ) \cdot 5(\cos 106^\circ + i \sin 106^\circ)$

18) $2(\cos 135^\circ + i \sin 135^\circ) \cdot 7(\cos 213^\circ + i \sin 213^\circ)$

19) $\dfrac{6(\cos 78^\circ + i \sin 78^\circ)}{2(\cos 27^\circ + i \sin 27^\circ)}$ 20) $\dfrac{3\sqrt{2}(\cos 238^\circ + i \sin 238^\circ)}{6(\cos 103^\circ + i \sin 103^\circ)}$

21) $\dfrac{1}{3(\cos 210^\circ + i \sin 210^\circ)}$ 22) $\dfrac{5(\cos 10^\circ + i \sin 10^\circ)}{2(\cos 209^\circ + i \sin 209^\circ)}$

---

[*] You may want to write programs using the verbal algorithms presented in this section.

23) $3\ (\cos 18^\circ + i \sin 18^\circ)^2$ 24) $5 \cos 250^\circ + i \sin 250^\circ)^2$

(25 - 28) Change the following complex numbers to polar form, perform the indicated operations in polar form and then express the results in rectangular form.

25) $(2 + 3i\sqrt{3})(2 - i\sqrt{3})$ 26) $(\sqrt{3} - i) \div (1 + i\sqrt{3})$

27) $(5 - 2i) \div (6 + .5i)$ 28) $(1 + 3i)^4$

29) Find the distance between the points $a + bi$ and $c + di$.

30) Show that if $r_1(\cos \theta_1 + i \sin \theta_1) = r_2(\cos \theta_2 + i \sin \theta_2)$

then $r_1 = r_2$ and $\theta_1 = \theta_2 + k\ 360^\circ$ where k is an integer.

7.5 De Moivre's Theorem

In section 7.4 we developed means for multiplying complex numbers in polar form. We can extend that idea to powers and roots.

Let $z = r(\cos\theta + i\sin\theta)$ and n a positive integer then

$$Z^n = \left[r\,(\cos\theta + i\sin\theta)\right] \cdot \left[r\,(\cos\theta + i\sin\theta)\right] \cdot \ldots \cdot$$

$$\left[r\,(\cos\theta + i\sin\theta)\right] \quad n \text{ terms}$$

$$= \underbrace{r\cdot r\cdot \ldots \cdot r}_{n\text{ terms}} \left[\cos\,(\underbrace{\theta+\theta+\ldots+\theta}_{n\text{ terms}}) + i\sin\,(\underbrace{\theta+\theta+\ldots+\theta}_{n\text{ terms}})\right]$$

$$= r^n\left[\cos\,(n\cdot\theta) + i\sin\,(n\cdot\theta)\right].$$

thus $\left[r\,(\cos\theta + i\sin\theta)\right]^n = r^n\left[\cos\,(n\cdot\theta) + i\sin\,(n\cdot\theta)\right].$

We have just proved a special case of a more general theorem known as DeMoivre's Theorem.

Theorem 7.5.1 (DeMoivre)

If n be any complex number then

$$\left[r\,(\cos\theta + i\sin\theta)\right]^n = r^n\left[\cos\,(n\cdot\theta) + i\sin\,(n\cdot\theta)\right].$$

DeMoivre's theorem is named after Abraham de Moivre (1667-1754). He was born in France but did most of his work in England where he was a friend of Newton and a fellow of the Royal Society.

Using DeMoivre's theorem we can not only find powers but roots of complex numbers.

EXAMPLE 7.5.2 Determine $(3 + 4i)^{10}$ using De Moivre's Theorem.

Solution: $3 + 4i \doteq 5(\cos 53.1301^\circ + \sin 53.1301^\circ)$

$$[5(\cos 53.1301^\circ + i \sin 53.1301^\circ)]^{10} = 5^{10}[\cos(10 \cdot 53.1301^\circ) + i \sin (10.53.1301^\circ)]$$

$$\doteq 9{,}765{,}625 \ [\cos (171.3010)^\circ + i \sin (171.3010)^\circ]$$

$$\doteq -9{,}653{,}286.999 + 1{,}476{,}984.007i$$

$$= -9{,}653{,}287 + 1{,}476{,}984i$$

EXAMPLE 7.5.3 Using DeMoivre's theorem find the three cube roots of 8.

Solution: $8 = 8 (\cos 0^\circ + i \sin 0^\circ)$ and

$$8 = 8 \ [\cos (0 + k \cdot 360)^\circ + i \sin (0 + h360)^\circ]$$

where k is any positive integer

$$8^{\frac{1}{3}} = 8^{\frac{1}{3}} \ \left[\cos\left(\frac{0+k360}{3}\right)^\circ + i \sin \left(\frac{0+k\cdot 360}{3}\right)^\circ\right]$$

$$= 2 \ \left[\cos\left(\frac{0}{3}\right)^\circ + i \sin\left(\frac{0}{3}\right)^\circ\right] \quad \text{when } k = 0$$

$$= 2 \ \left[\cos\left(\frac{360}{3}\right)^\circ + \sin\left(\frac{360}{3}\right)^\circ\right] \quad \text{when } k = 1$$

$$= 2 \ \left[\cos\left(\frac{720}{3}\right)^\circ + \sin\left(\frac{720}{3}\right)^\circ\right] \quad \text{when } k = 2$$

If $k = 3$ the result is the same as for $k = 0$.

$2(\cos 0^\circ + i \sin 0^\circ) = 2$

$2(\cos 120^\circ + i \sin 120^\circ) = -1 + i\sqrt{3}$

$2(\cos 240^\circ + i \sin 240^\circ) = -1 - i\sqrt{3}$

The cube roots of 8 are 2, $-1 + i\sqrt{3}$, $-1 - i\sqrt{3}$.

It can be proved, by means beyond our present grasp, that:

$$e^{\theta} = 1 + \frac{\theta}{1} + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \ldots$$

$$\sin\theta = \frac{\theta}{1} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \ldots$$

where e is the base of natural logarithms and θ is measured in radians.

If we let $i\cdot\phi = \theta$ where $i = \sqrt{-1}$ we obtain

$$\cos\theta = 1 - \frac{(i\phi)^2}{2!} + \frac{(i\phi)^4}{4!} - \frac{(i\phi)^6}{6!} + \frac{(i\phi)^8}{8!} + \ldots$$

$$= 1 + \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \frac{\phi^6}{6!} + \frac{\phi^8}{8!} + \ldots$$

$$\sin\theta = \frac{i\phi}{1} - \frac{(i\phi)^3}{3!} + \frac{(i\phi)^5}{5!} - \frac{(i\phi)^7}{7!} + \frac{(i\phi)^9}{9!} - \ldots$$

$$= i\left(\frac{\phi}{1} - \frac{i^2\phi^3}{3!} + \frac{i^4\phi^5}{5!} - \frac{i^6\phi^7}{7!} + \frac{i^8\phi^9}{9!} - \ldots\right)$$

$$\frac{\sin\theta}{i} = \frac{\phi}{1} + \frac{\phi^3}{3!} + \frac{\phi^5}{5!} + \frac{\phi^7}{7!} + \frac{\phi^9}{9!} + \ldots$$

$$\frac{\sin\theta}{i}\cdot\frac{i}{i} = \frac{i\sin}{-1}\cdot\frac{-1}{-1} = -i\sin\theta$$

$$\cos\theta - i\sin\theta = 1 + \frac{\phi}{1} + \frac{\phi^2}{2!} + \frac{\phi^3}{3!} + \frac{\phi^4}{4!} + \ldots .$$

$$\cos\theta - i\sin\theta = e^{\phi} = e^{\frac{\theta}{i}} = e^{-i\theta}$$

now replace θ by $-\theta$, $\cos\theta = \cos(-\theta)$

$-\sin\theta = \sin(-\theta)$

and we obtain

$$\boxed{\cos\theta + i\sin\theta = e^{i\theta} .}$$

This important result is called Euler's formula. It has particular significance to electrical and electronic engineers. If $\theta = \pi$ we have

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) \quad \text{or}$$

$$\text{Euler's Equation:} \quad e^{i\pi} = -1$$

Exercise Set 7.5

(1 - 8) Use DeMoivre's Theorem to find the value of each of the following expressions.

1) $(-3 + i)^4$
2) $(2 - 3i)^5$
3) $[3 (\cos 28^\circ + i \sin 28^\circ)]^5$
4) $[2 (\cos 107^\circ + i \sin 107^\circ)]^7$
5) $(1 - i\sqrt{2})^{10}$
6) $(\sqrt{3} - i)^7$
7) $(-1 -i)^{-8}$
8) $(-1 +i)^{-6}$

9) Find the fourth roots of $(2 - 2i\sqrt{3})$ and represent your answers in algebraic form.

10) Find the cube roots of $(-5 + 3i)$ and represent your answers in algebraic form.

11) Find the cube roots of 1.

12) Find the fourth roots of i.

(13 - 16) Any polynomial of degree n with complex numbers as coefficients has exactly n distinct complex roots. Find all the roots of each of the following equations.

13) $x^4 - 1 = 0$ 14) $x^5 - 32 = 0$

15) $x^3 + 27i = 0$ 16) $x^2 = i\sqrt{2} - 1$

17) Use Euler's formula to verify DeMoivre's theorem for any positive integer n.

(18 - 19) Use Euler's formula to show that

18) $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ 19) $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

20) Explain why a positive integer raised to an irrational power as $2^{\sqrt{2}}$ has an infinite number of values none of which is a real number.

7.6 Calculating Trigonometric Functions

You have always determined specific values of trigonometric functions by either using tables or pushing keys on your calculating device. But where do these numbers come from?

One way to compute trigonometric values is by using infinite series. Recall that a polynomial, P(x), is an expression of the form $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2} + x^{n-2} + \ldots + a_2x^2 + a_1x^1 + a_0$ where the coefficients are constants and n is a positive integer. If a polynomial continued forever it would be a power series.

Definition 7.6.1

The expression $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$ where the c_i's are constants is called a power series.

A polynomial has a finite number of terms. A power series has infinitely many terms. Thus, when we evaluate a power series we can only calculate an approximate value.

EXAMPLE 7.6.2 Evaluate $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{5!} + \frac{x^6}{6!} + \ldots$ when $x = .5$ using only the first six terms of the series.

$$f(.5) \doteq 1 + .5 + \frac{.5^2}{2!} + \frac{.5^3}{3!} + \frac{.5^4}{4!} + \frac{.5^5}{5!} + \frac{.5^6}{6!}$$

$$\doteq 1 + .5 + \frac{.25}{2} + \frac{.125}{6} + \frac{.0625}{24} + \frac{.03125}{120} + \frac{.015625}{720}$$

$$\doteq 1.648719618$$

Notice that the values of higher degree terms get smaller because the numerators get smaller and the denominators get larger. If $x = 1$ this series becomes

$$f(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots = e$$

This particular power series is an "old friend."

Definition 7.6.3

$$e^x = f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

other "old friends" that can be represented as power series are:

Definition 7.6.4

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots; \quad \theta \text{ in radians}$$

Definition 7.6.5

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots; \quad \theta \text{ in radians}$$

These particular power series are of a generic type called Taylor series. When you study the Calculus you will not only study why these series represent these functions but other properties and applications of these series.

EXAMPLE 7.6.6 Calculate $\cos(\frac{\pi}{2})$ using the first five terms of its power series.

Solution:

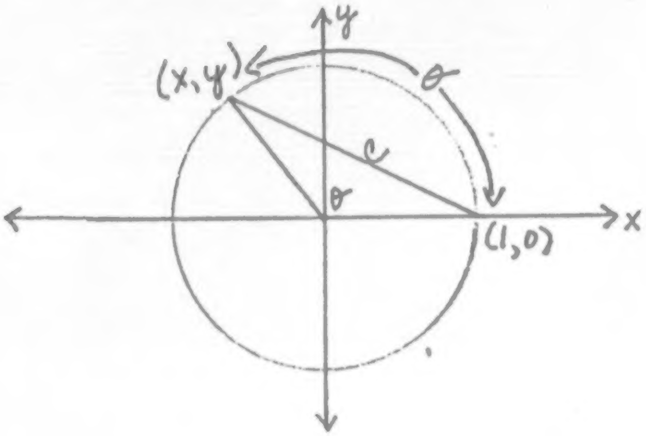
$$\cos(\tfrac{\pi}{4}) = 1 - \frac{(\frac{\pi}{4})^2}{2!} + \frac{(\frac{\pi}{4})^4}{4!} - \frac{(\frac{\pi}{4})^6}{6!} + \frac{(\frac{\pi}{4})^8}{8!} - \ldots$$

$$\doteq 1 - \frac{(\frac{\pi}{4})^2}{2} + \frac{(\frac{\pi}{4})^4}{24} - \frac{(\frac{\pi}{4})^6}{720} + \frac{(\frac{\pi}{4})^8}{40320} .$$

$$\doteq .707106806$$

We know that $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \doteq 707106781$ so it is easy to see that only five terms of this power series give accuracy up to six decimal places.

Another way to calculate values for trigonometric series is to consider the geometry of the unit circle and use recursion. Let θ be a measure of an angle in radians. When θ is placed in the unit circle we have the following diagram.



We shall be interested in the length c of the chord of arc length θ determined by the angle of measure θ .

$x^2 + y^2 = 1$ (x,y) is on the unit circle

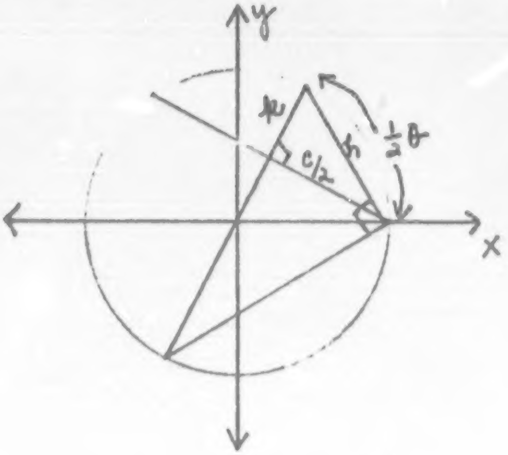
$(x-1)^2 + (y-0)^2 = c^2$ distance formula

By subtracting the second equation from the first we obtain

$$2x - 1 = 1 - c^2$$

so $\cos \quad = x = a - \frac{1}{2}c^2$

The diameter perpendicular to our chord bisects it and its arc. Also, the angle inscribed in the semicircle is a right angle.



and we have similar triangles. The diameter is of length 2 so $k:h = h:2$ and $k = \frac{h^2}{2}$. Also, by the Pythagorean theorem

$$k^2 + \frac{c^2}{4} = h^2$$

thus $$\left(\frac{h^2}{2}\right)^2 + \frac{c^2}{4} = h^2$$

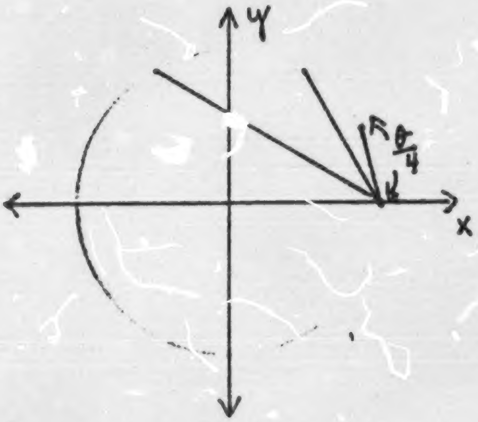
or $$c^2 = h^2(4 - h^2).$$

The important thing to notice in this formula is that if A_1 represents h^2, the square of the chord for the arc $\theta/2$, then $A_1(4-A_1)$ gives the square of the chord for the arc θ. As we saw above, cos θ is 1 minus half this number.

How can we find A_1? Since $\frac{\theta}{4} = (\frac{1}{2})(\frac{\theta}{2})$,

$$A_1 = A_2(4-A_2)$$

where A_2 is the square of the chord for $\frac{\theta}{4}$.



Similarly, $A_2 = A_3(4-A_3)$, A_3 being the square of the chord for $\theta/8$. In general

$$A_{k-1} = A_k(4-A_k)$$ where A_k is the square of the chord for $\theta/2^k$.

If we are actually doing the computations this way we must start somewhere. If we are given θ between 0 and 2π, it turns out that arc $\frac{\theta}{2^{16}} = \frac{\theta}{65536}$ is so small that it is "very nearly equal" to its own chord. Thus a very small error is made in using $(\theta/65536)^2$ for the $(\text{chord})^2$.

We can present this calculation as a verbal algorithm.

<u>7.6.6</u> Verbal algorithm for computing $\cos\theta$ where $0 < \theta < 2\pi$.

1. $A \leftarrow (\theta/65536)^2$, $N \leftarrow 0$
2. $A \leftarrow A(4 - A)$
3. $N \leftarrow N + 1$
4. If $N = 16$ display $1 - 0.5A$ and STOP otherwise go to step 2.

EXAMPLE 7.6.7 Write a program for your calculating device for verbal algorithm 7.6.6.

<u>HP 33E</u>

```
01    ENTER              12    STO 2           23    RCL 3
02    f clear reg        13     1              24    RCL 2
03     6                 14     6              25    f  x ≠ y?
04     5                 15    STO 3           26    GTO 16
05     5                 16    RCL 1           27    RCL 1
06     3                 17    CHS             28     .
07     6                 18     4              29     5
08     ÷                 19     +              30     x
09    g x²               20    STO x 1         31    CHS
10    STO 1              21     1              32     1
11     0                 22    STO + 2         33     +
```

TI-58

```
00     ÷          16   2nd Lbl          32   x ≠ t?
01     6          17     A              33    A
02     5          18   RCL              34   RCL
03     5          19    01              35    01
04     3          20   +/-              36    ×
05     6          21    +               37    .
06     =          22    4               38    5
07    x^2         23    =               39    =
08   STO          24   2nd Prd          40   +/-
09    01          25    01              41    +
10     0          26    1               42    1
11   STO          27   SUM              43    =
12    02          28    02              44   R/S
13     1          29   RCL              45   RST
14     6          30    02
15   x ⇄ t        31   INV 2nd
```

TRS-80

```
      10   INPUT T
      20   A = (T/65536) ↑ 2
      30   N = 0
      40   A = A(4 - A)
      50   N = N + 1
      60   If N = 16  PRINT 1 - 0.5*A AND STOP
             ELSE 20.
```

Exercise Set 7.6

(1 - 6) Using the first seven terms of the power series expansion of e^x determine each of the following. Compare your results with those obtained by directly using the key (or instruction) for powers of e on your calculating device.*

1) e^2 2) e^3

3) $e^{.3}$ 4) $e^{.6}$

---

*You may wish to write a program to do these exercises.

5) e^{-2} 6) e^{-1}

(7 - 14) Using the first five terms of the power series expansion of $\cos\theta$ or $\sin\theta$ determine each of the following. Compare your results with those obtained by directly using the key (or instruction) for these functions on your calculating device.*

7) $\sin \pi$ 8) $\cos \pi$

9) $\cos \pi/6$ 10) $\sin \frac{\pi}{2}$

11) $\cos(-\pi)$ 12) $\sin(-\pi)$

13) $\sin 27^\circ$ 14) $\cos 123^\circ$

15) To how many decimal places would you expect the approximation of e^2 to be accurate if you use only the first seven terms of its power series? Explain. (Confer exercise 1.)

16) How many decimal places would you expect the approximation of $\cos \frac{\pi}{6}$ to be accurate if you use only the first five terms of its power series? Explain. (Confer exercise 8.)

17) Plot a graph of $y = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for $-4 \leq x \leq 4$ at intervals of .5. On the same set of axes plot y = sin x. For what values of x do the graphs seem to be close together?

18) Plot a graph of $y = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ for $-4 \leq x \leq 4$ at intervals of .5. On the same set of axes plot y = cos x. For what values of x do the graphs seem to be close together?

(19 - 22) Use algorithm 7.6.6 (or programs 7.6.7) to determine each of the following. Compare your results to the indicated

---

*You may wish to write a program to do these exercises.

19) $\cos \pi/6$ (9)

20) $\cos \pi$ (18)

21) $\cos(-\pi)$ (11)

22) $\cos 123^\circ$ (14)

Chapter 7 TEST

1) Find the rectangular coordinates of the following point: $(3\sqrt{2}, 45^\circ)$.

2) Represent the point $(-2\sqrt{3}, 2)$ in polar coordinate form with $r < 0$.

3) Transform the equation $x + y = 6$ from rectangular to polar coordinates.

4) Transform the equation $\rho^2 + 3\rho \cos \theta = 4$ from polar to rectangular coordinates.

5) Express $2(\cos 210^\circ + i \sin 210^\circ)$ in $a + bi$ form.

6) Change $x^2 + y^2 = 2x + 15$ from rectangular coordinates to polar coordinates.

7) Express $1 - i$ in the form $r(\cos \theta + i \sin \theta)$.

8) Express in the form $a + bi$ the quotient obtained when $12(\cos 90^\circ + i \sin 90^\circ)$ is divided by $3(\cos 60^\circ + i \sin 60^\circ)$.

9) A root of $x^3 + 8 = 0$ lies in quadrant IV. Express this root in the form $r(\cos \theta + i \sin \theta)$.

10) Multiply $2(\cos 115^\circ + i \sin 115^\circ)$ by $6(\cos 245^\circ + i \sin 245^\circ)$ and express the result in the rectangular form, $a + bi$.

11) A complex root of $x^5 + 32 = 0$ is:

    (1) $2(\cos 72^\circ + i \sin 72^\circ)$
    (2) $2(\cos 36^\circ + i \sin 36^\circ)$
    (3) $-2(\cos 108^\circ + i \sin 108^\circ)$
    (4) $-2(\cos 36^\circ + i \sin 36^\circ)$

12) Express the product of $3 + 2i$ and $2i$ in the form $r \text{ cis } \theta$.

13) Write the expression $[4(\cos 30^\circ + i \sin 30^\circ)]^2$ in the form $a + bi$.

14) Express the complex number $\sqrt{2} + i\sqrt{2}$ in the polar form $r(\cos \theta + i \sin \theta)$.

15) Express the quotient $\frac{2 + 3i}{1 - 3i}$ in polar form. (Simplify completely.)

16) Using DeMoivre's Theorem find:

   (a) $[2(\cos 45^\circ + i \sin 45^\circ)]^5$

   (b) $(1 + \sqrt{3})^3$

17) Using De Moivre's Theorem, find the indicated roots:

(a) The five fifth roots of $\sqrt{2} - i\sqrt{2}$

(b) The four fourth roots of $2(\cos 100^\circ + i \sin 100^\circ)$

18) Complete the table and then, on the Polar coordinate paper provided, sketch the graph of $r = 1 - 2\cos\theta$

Fix 1 on your calculator

θ	r	θ	r	θ	r
10°		130°		250°	
20°		140°		260°	
30°		150°		270°	
40°		160°		280°	
50°		170°		290°	
60°		180°		300°	
70°		190°		310°	
80°		200°		320°	
90°		210°		330°	
100°		220°		340°	
110°		230°		350°	
120°		240°		360°	